ALTERNATING CURRENTS
AND TRANSIENTS

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TREATED BY THE ROTATING VECTOR METHOD


BY

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# ALTERNATING CURRENTS AND TRANSIENTS

## TREATED BY THE ROTATING-VECTOR METHOD

### INTRODUCTION

**1. The Object and Scope of the Book.**—The present work, though not a textbook of general electrical theory, is intended for the use of students of electricity, more particularly for students of applied electricity in the domain of light and heavy electrical engineering. Its object is to provide such students with a useful mathematical equipment for the solution of many of the theoretical and practical problems likely to be encountered in the study and the practice of this subject.

The method considered is that which has hitherto been variously described as the "Symbolic Method," "Complex Algebra," or the "Rotating-vector Method."

In the first part of the book this method is developed on the basis of the simplest possible definitions of the quantities involved, its essentially vectorial character is elucidated, and, in addition, certain fundamental propositions are established by means of which the range of the applicability of the method is extended so as to include the analysis of transient alternating phenomena.

The remainder of the book is devoted to the practice of the method, illustrated by application to certain typical branches of electrical theory. The treatment of such typical subjects is not intended to be detailed or complete. They are presented in outline only, in order that the application of the method may be illustrated over as wide a range as possible without obscuring the main object of the book by excess of detail.

**2. The Necessity for a Work of This Character.**—It may be of interest to relate how this book originated. As a student of

electrical engineering, the author had long been familiar with the rotating-vector method of representation of alternating-current relationships, and had fully appreciated its many and important advantages over any purely algebraic method of analysis. During the same period he was studying applied mathematics and electromagnetic theory, and, in connection with these subjects, became acquainted with pure vector analysis as developed by Heaviside, Gibbs, and others. He soon noted that, while many fundamental ideas were common to both these applications of vectorial notation, there were some apparent differences and even inconsistencies. For instance, in connection with alternating-current theory the symbol "j" was defined as being the imaginary quantity $\sqrt{-1}$, and expressions such as (a + jb), (R + jX), etc. (a, b, R, X being scalar quantities) were variously described as complex numbers or vectors. In so far as the quantity (a + jb) could be represented by a line having a definite magnitude and direction, the term "vector" seemed an appropriate description of it, since magnitude and direction, taken together, are the essential characteristics of a vector. A difficulty arose, however, in the application of this notation to telephone and telegraph transmission problems, where expressions such as (a + jb) (c + jd) are encountered. If each of these factors is properly described as a vector, the product of the two is presumably some form of vector product. In the system of pure vector analysis, two types of vector product are recognized, the scalar product of vectors and the vector product of vectors. The expression (a + jb) (c + jd) did not appear to fall into either category. Was this, then, a third form of vector product?

The persons to whom the author put this question agreed as to the distinction between this expression and the usual forms of vector product, but they were unable to resolve the apparent inconsistency in nomenclature and interpretation. This, however, the author was eventually able to do for himself, for a careful consideration of the fundamental ideas involved soon showed that, in relation to the rotating-vector method of analysis, expressions such as (a + jb), (R + jX) etc. are not properly described either as vectors or as complex numbers. Their true character is that of "versor operators," having a certain definite and calculable effect on any vector operand with which they are associated. In the representation of such quantities by means of lines of definite direction and magnitude, it will be found that

some unit vector operand has been implicitly assumed, usually a unit of length in the direction of the "x" axis of coordinates.

By making this unit vector operand explicit instead of implicit, it was found that the nomenclature of pure vector analysis and of the rotating-vector method in alternating-current theory could be brought into perfect harmony.

**3. The Symbol "j."**—It has always seemed to the author that the interpretation of the symbol "j" in the rotating-vector method as the imaginary quantity $\sqrt{-1}$ is neither necessary nor desirable, and is due more to the historical development of the subject than to its logical requirements.

His experience has shown that the introduction of the terms "imaginary" or "complex" into what are essentially real and simple operations is a definite hindrance rather than a help to the student when he first makes the acquaintance of this method of analysis.

The student eventually realizes that all he need understand of the symbol "j" is that it rotates a vector through 90° in a certain defined direction, and thereafter the terms "imaginary" and "complex" cease to have any other meaning in this connection and so lose their power to mystify his intelligence.

This being the case, is it not simpler to make this final conception of the symbol also the initial one; to define it, that is to say, as an operator having a certain defined effect on any vector operand with which it is associated? On the basis of this simple and easily comprehended definition the whole structure of the rotating-vector method, including all the necessary operations of what is known in pure mathematics as "complex algebra," can be built up into a logical and self-consistent whole.

It is on the above principles that the author has compiled the present account of the subject, and he hopes that in doing so he has succeeded in clearing away for other students some of the difficulties which he himself encountered in the earlier period of his studies.

## CHAPTER I

## VECTORS AND VECTOR OPERATORS

**1. The Definition of a Unit Vector.**[1]—*A unit vector is a unit of length measured in some definite direction in space.*

Such a quantity can be represented by a line of unit length measured in the given direction. It is usual to terminate the line with an arrow head to distinguish between the initial and the terminal points of the line. Thus, the lines AB, CD, EF, and GH in Fig. 1 can be considered to represent the unit vectors $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, and $\mathbf{d}_1$ respectively.

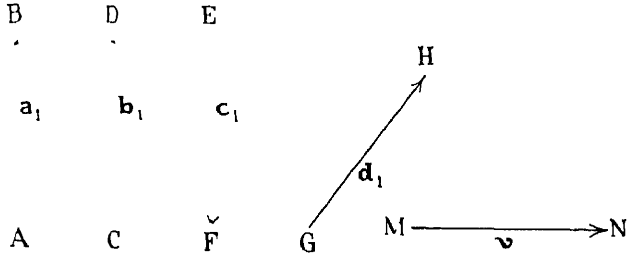


FIG. 1.

It is important to realize that the essential element in the above definition is the idea of direction as distinct from position. Thus, AB and CD considered as unit vectors are identical, since they have the same direction, *i.e.*,

$$\mathbf{a}_1 = \mathbf{b}_1. \tag{1}$$

On the other hand, CD, though equal to EF in magnitude, is opposite to it in direction, therefore,

$$\mathbf{b}_1 \neq \mathbf{c}_1, \tag{2}$$

though, since the lines which represent $\mathbf{b}_1$ and $\mathbf{c}_1$ are parallel, there is a relation between these two unit vectors which will be considered later.

Again, GH is equal to any of the other lines in magnitude, but the unit vector $\mathbf{d}_1$ which it represents is unequal to $\mathbf{a}_1$, $\mathbf{b}_1$, or $\mathbf{c}_1$ since it differs from them all in direction.

[1] Bibliography, No. 22.

For convenience of reference, the symbol $\mathbf{v}$ will be used throughout to denote the unit vector whose direction lies in the plane of the paper and is parallel to the bottom edge of the page measured from left to right. It is represented by the line MN in Fig. 1.

**2. Derivation of a Vector from a Unit Vector.**—Starting with some unit vector, say, $\mathbf{v}$, its length can be extended a times without altering its direction, a being any positive real number. The resulting quantity can be represented by a line a units in length drawn in the direction of $\mathbf{v}$ (Fig. 2).

Such a quantity is termed a "vector." The term can equally well be applied to any line such as AB (Fig. 2) which is used to represent the vector, since in the true sense of the word a

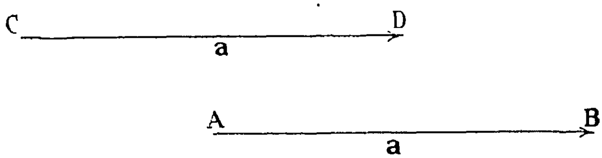


Fig. 2.

vector is not confined to any definite position in space, but only to some definite direction in space. Thus, the lines AB and CD in Fig. 2 are both the vector $\mathbf{a}$.

The relation of $\mathbf{a}$ to $\mathbf{v}$ can be expressed by the equation

$$\mathbf{a} = a\mathbf{v}, \tag{3}$$

which means that the vector $\mathbf{a}$ is in the direction of $\mathbf{v}$ and that its magnitude is a times the magnitude of $\mathbf{v}$. Another way of considering the matter is to regard a as an operator the effect of which is to increase the length of $\mathbf{v}$ a times without altering its direction.

It is clear that any vector whatever in the direction of $\mathbf{v}$ can be represented as $\mathbf{v}$ multiplied by some scalar number. In general any vector can be similarly expressed in terms of a scalar number which defines its magnitude and some unit vector which defines its direction.

**3. The Rotation of a Vector in a Given Plane.**—In the preceding paragraph it was seen that the scalar coefficient a (Eq. (3)) could be regarded as an operator, the effect of which was to increase the magnitude of its operand, in this case the unit vector $\mathbf{v}$, a times, without altering its direction. The natural comple-

ment of such an operator is one whose effect is to alter the direction of its operand without altering its magnitude. Such an operator will, therefore, be defined in the following terms:

*The operator* j *is one whose effect is to rotate its operand through 90° in a definite sense in some given plane, without altering its magnitude.*

In all that follows the given plane will be taken to be the plane of the paper, and the sense will be taken to be counterclockwise, *i.e.*, the operator j is one, the effect of which is to rotate its operand through 90° in a counterclockwise sense in the plane of the paper without altering its magnitude, it being understood that its operand is always some vector or unit vector whose direction lies in the plane of the paper.

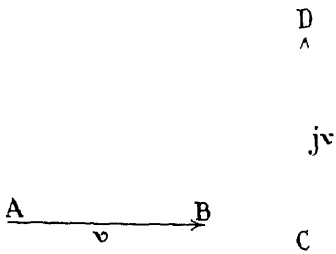


FIG. 3.

Thus, in Fig. 3, since the unit vector **v** is represented by AB, then j**v** will be represented by CD.

**4. Vector Addition.**—Given two vectors **a** and **b** (Fig. 4) represented by the lines AB and CD, what is to be understood by the sum (**a** + **b**)?

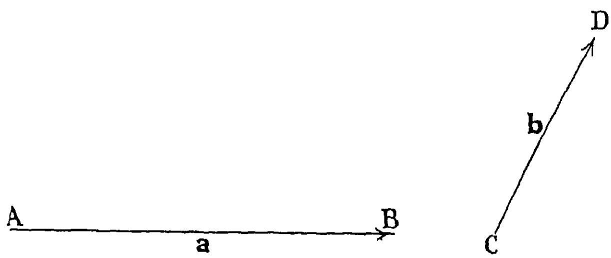


FIG. 4.

The idea is most easily appreciated by regarding the addition as the going of a certain distance a in one direction, followed by the going a certain distance b in another direction. The result will be a displacement of magnitude, say r, in some new direction which will, in general, be different from either of the other two. The process can be represented graphically as shown in Fig. 5. It is in this sense that the vector **r** is said to be the sum of **a** and **b**, *i.e.*,

$$\mathbf{r} = \mathbf{a} + \mathbf{b}. \tag{4}$$

It should be noted that the result of adding two vectors is independent of the order in which the process is performed. Thus, **r** in Fig. 5, which represents the addition of **b** to **a**, is

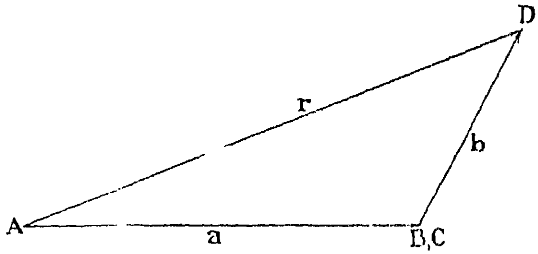


FIG. 5.

dentical with **r** in Fig. 6, which represents the addition of ι to **b**, *i.e.*,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \tag{5}$$

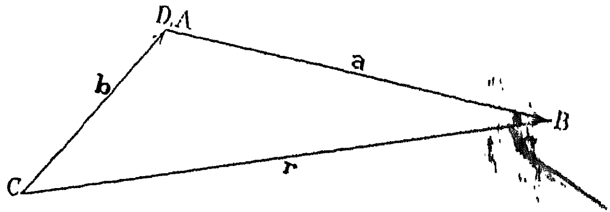


FIG. 6.

**5. The Significance of the Negative Sign as Applied to Vec- rs.**—In view of the above (Par. 4) it is obvious that, for the iit vectors **b** and **c** illustrated in Fig. 1,

$$\mathbf{b} + \mathbf{c} = 0 \tag{6}$$

$$\mathbf{b} = -\mathbf{c}, \tag{7}$$

Thus, if two unit vectors (or vectors, by a simple extension of ; same reasoning) are related as shown in Eqs. (6) or (7), then ;y are equal in magnitude but opposite in direction. Another y of expressing the same thing is that *the operator* $-1$ *reverses direction of its operand.*

**ι. The Subtraction of Vectors.**—An immediate application of above (Par. 5) is that the subtraction of one vector from ther is essentially of the same nature as addition, *i.e.*,

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}), \tag{8}$$

in other words, to subtract **b** from **a** it is only necessary to reverse the direction of **b** and then add it to **a**. The process is illustrated in Fig. 7.

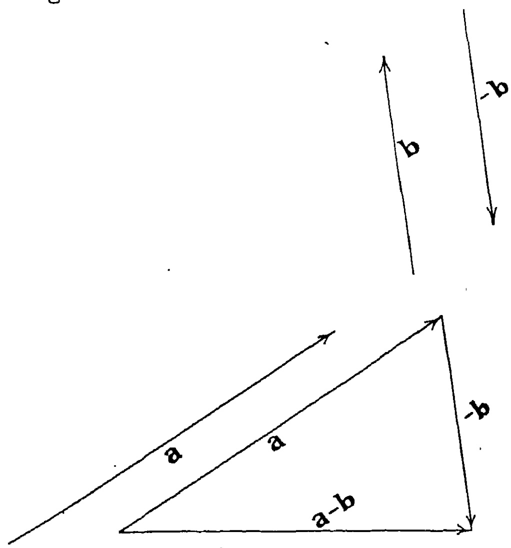


FIG. 7.

It should be noted that

$$\mathbf{a} - \mathbf{b} = -(\mathbf{b} - \mathbf{a}). \tag{9}$$

This is illustrated in Fig. 8, and is, of course, in accordance with the usual rules relating to the negative sign.

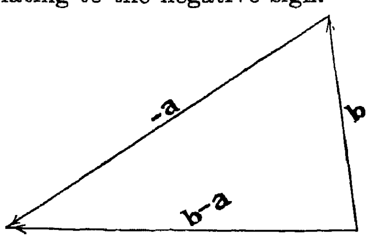


FIG. 8.

**7. The Addition and Subtraction of Any Number of Vectors.**—The processes illustrated in Pars. 5 and 6 can obviously be extended to any number of vectors. Thus, in Figs. 9 and 10

$$\mathbf{r}_1 = \mathbf{a} + \mathbf{b} + \mathbf{c} \tag{10}$$

$$\mathbf{r}_2 = \mathbf{a} + \mathbf{b} - \mathbf{c}. \tag{11}$$

Further, since

$$\mathbf{b} + \mathbf{c} = \mathbf{c} + \mathbf{b} \tag{12}$$

then

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{a} + \mathbf{c} + \mathbf{b} \tag{13}$$

and, since any order of the letters can be obtained by interchanging them two at a time, it follows from Par. 4 that the result of adding any number of vectors is independent of the order in

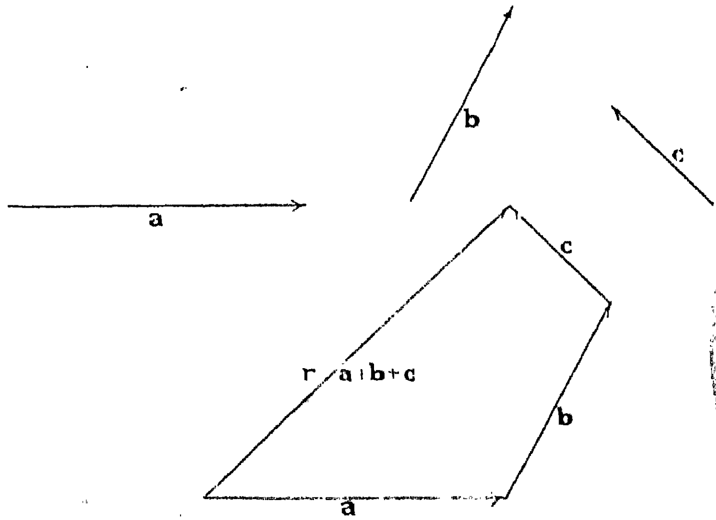


FIG. 9.

which the addition is carried out. For the reasons already given, the corresponding case of subtraction does not need separate consideration.

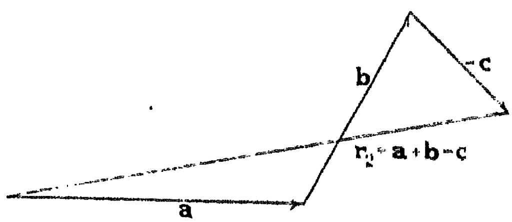


FIG. 10.

**8. The Vector,** $a\mathbf{v} + jb\mathbf{v}$.—It has been seen that $a\mathbf{v}$ is a vector of length a units and of direction parallel to that of $\mathbf{v}$. The vector $j\mathbf{v}$ is of unit length and is in a direction perpendicular to that of $\mathbf{v}$ in the plane of the paper. Thus, $bj\mathbf{v}$ or $jb\mathbf{v}$ is a vector b units in length and of direction perpendicular to that of $\mathbf{v}$. The addition of these two vectors in the manner described in Par. 4 will give rise to a new vector differing in direction and in magnitude from both $a\mathbf{v}$ and $jb\mathbf{v}$. In the above a and b can be any positive or

negative real numbers. The addition of the two vectors is illustrated in Fig. 11.

Since the angle $A\hat{B}C$ is a right angle, the magnitude r of AC is given by

$$r^2 = a^2 + b^2 \tag{14}$$

or

$$r = \sqrt{a^2 + b^2} \tag{15}$$

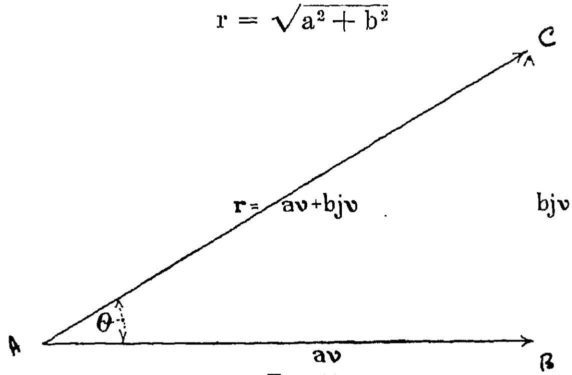


FIG. 11.

and $\theta$, the angle between $\mathbf{r}$ and $\mathbf{v}$, by

$$\tan\theta = \frac{b}{a}. \tag{16}$$

Thus, putting

$$a\mathbf{v} + jb\mathbf{v} = \mathbf{r}, \tag{17}$$

$\mathbf{r}$ is seen to be a vector of length

$$r = \sqrt{a^2 + b^2} \tag{18}$$

whose direction makes with that of $\mathbf{v}$ an angle

$$\theta = \tan^{-1}\frac{b}{a}. \tag{19}$$

**9. The Operator** (a + jb).—The two operations involved in the preceding paragraph (*i.e.*, multiplication by a, plus the result of multiplication by b combined with a positive rotation through 90°) can conveniently be grouped together into one expression, since both have the same operand, *i.e.*,

$$a\mathbf{v} + jb\mathbf{v} = (a + jb)\mathbf{v} = \mathbf{r}. \tag{20}$$

Thus, by operating on $\mathbf{v}$ with $(a + jb)$ it has been turned into a vector $\mathbf{r}$ of magnitude $(a^2 + b^2)^{\frac{1}{2}}$ and of slope relative to $\mathbf{v}$ given by $\theta = \tan^{-1}\frac{b}{a}$. In other words, the effect of the operator $(a + jb)$ has been to multiply the magnitude of $\mathbf{v}$ by $r = (a^2 + b^2)^{\frac{1}{2}}$ and to rotate it through an angle $\theta$.

These quantities r and $\theta$ depend only on a and b, and not on $\mathbf{v}$. That is to say, *the effect of the operator* $(a + jb)$ *is independent of the magnitude and direction of its operand.* It may, therefore, be said that the effect of the operator $(a + jb)$ is to multiply the magnitude of its operand by $r = (a^2 + b^2)^{\frac{1}{2}}$ and to rotate it through an angle $\tan^{-1}\dfrac{b}{a}$ in the plane of the paper in a positive (anticlockwise) sense. Thus, referring to Fig. 12, if $\mathbf{z}$ be a vector of magnitude z, whose direction makes with that of $\mathbf{v}$ an angle $\phi$, then $(a + jb)\mathbf{z}$ is a vector of magnitude rz making with $\mathbf{v}$ an angle $(\phi + \theta)$.

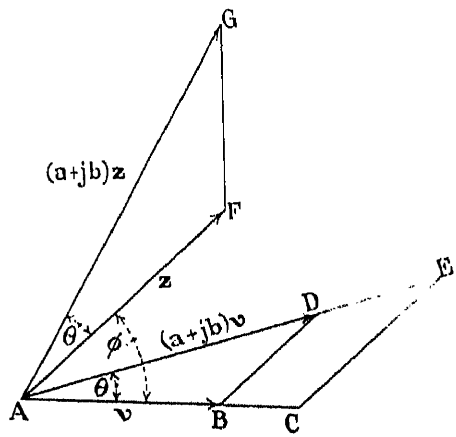


FIG. 12.

**10. Alternative Form for** $(a + jb)$.—Referring to Fig. 11 (Par. 8)

$$\frac{AB}{AC} = \frac{a}{r} = \cos\theta \tag{21}$$

$$\therefore \quad a = r\cos\theta. \tag{22}$$

Similarly,

$$b = r\sin\theta \tag{23}$$

$$\therefore \quad (a + jb) = r(\cos\theta + j\sin\theta). \tag{24}$$

Thus, any operator of the form $(a + jb)$ can be put in the form $r(\cos\theta + j\sin\theta)$, where

$$r = \sqrt{a^2 + b^2} \tag{25}$$

$$\theta = \tan^{-1}\frac{b}{a}. \tag{26}$$

**11. The Operator** $(\cos\theta + j\sin\theta)$.—Since the effect of operating on any vector with $r(\cos\theta + j\sin\theta)$ is to multiply its magni-

tude by r and to rotate it through an angle $\theta$, it is clear that the effect of the operator $\cos\theta + j\sin\theta$ alone is to rotate the operand through an angle $\theta$ without altering its magnitude.

Thus, the operations represented by r and $(\cos\theta + j\sin\theta)$ are complementary to each other, since the one alters magnitude without altering direction, and the other alters direction without altering magnitude. The former is sometimes called a "tensor operator" and the latter a "versor operator."

**12. Alternative Form of Expression for** $(\cos\theta + j\sin\theta)$.— It will be convenient to find a more compact form of expression for the operator $(\cos\theta + j\sin\theta)$. For this purpose the nature of the operator j must be considered a little more closely.

From the definition of j

$$jj\mathbf{v} = -\mathbf{v} \tag{27}$$

$$jjj\mathbf{v} = -j\mathbf{v} \tag{28}$$

$$jjjj\mathbf{v} = \mathbf{v}. \tag{29}$$

If by analogy with the notation of scalar quantities, successive operations with j be denoted by powers of j, the above becomes

$$j^2\nu = -\nu \tag{30}$$

$$j^3\nu = -j\nu \tag{31}$$

$$j^4\nu = \nu \tag{32}$$

$$\therefore\ j^2 = -1 \tag{33}$$

$$j^3 = -j \tag{34}$$

$$j^4 = 1. \tag{35}$$

Thus, powers of j are related to unity and to each other in the same way as corresponding powers of $\sqrt{-1}$. It should be borne in mind, however, that this fact does not necessarily establish any identity between j and $\sqrt{-1}$. The symbol j represents an entirely real operation, whereas $\sqrt{-1}$ is an "imaginary quantity." It is enough for the present purposes that powers of j and powers of $\sqrt{-1}$ have similar interrelationships.

Returning now to the operator $(\cos\theta + j\sin\theta)$ and utilizing the series form for the sine and the cosine of a real angle, *i.e.*,[1]

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \textit{etc., etc. ad. inf.} \tag{36}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \textit{etc., etc. ad. inf.,} \tag{37}$$

then

$$(\cos\theta + j\sin\theta) = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \textit{etc., etc. ad. inf.} \tag{38}$$

[1] Bibliography, No. 23.

In virtue of the properties of powers of the operator j, this can be written

$$(\cos\theta + \mathrm{j}\sin\theta) = 1 + \mathrm{j}\theta + \frac{(\mathrm{j}\theta)^2}{2!} + \frac{(\mathrm{j}\theta)^3}{3!} + \textit{etc., etc. ad. inf.} \quad (39)$$

By analogy with the series form of $\epsilon^x$, *i.e.*,[1]

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad . \quad . \quad . \quad \textit{etc., etc. ad. inf.}, \quad (40)$$

the series above can be denoted by

$$(\cos\theta + \mathrm{j}\sin\theta) = \epsilon^{\mathrm{j}\theta}. \quad (41)$$

It must not be assumed, however, without further consideration that the $\mathrm{j}\theta$ in the above expression, *i.e.*, $\epsilon^{\mathrm{j}\theta}$, is really of the nature of an index, and subject, therefore, to the ordinary laws of indices. At present $\epsilon^{\mathrm{j}\theta}$ must be regarded as simply a short way of writing the series of Eq. (39) though the form it has taken suggests that the $\mathrm{j}\theta$ actually is of the nature of an index and has the corresponding properties.

**13. Application of the Laws of Indices to Operators of the Form $\epsilon^{\mathrm{j}\theta}$.**—A little consideration will show that the $\mathrm{j}\theta$ in operational expressions of the form $\epsilon^{\mathrm{j}\theta}$ does, in fact, obey the laws of indices.

Consider, for instance, the result of successive operations on a vector **r** with $(\cos\theta_1 + \mathrm{j}\sin\theta_1)$ and $(\cos\theta_2 + \mathrm{j}\sin\theta_2)$, *i.e.*,

$$(\cos\theta_1 + \mathrm{j}\sin\theta_1)(\cos\theta_2 + \mathrm{j}\sin\theta_2)\mathbf{r}. \quad (42)$$

Operation with $(\cos\theta_2 + \mathrm{j}\sin\theta_2)$ will produce a rotation $\theta_2$. Operating on the new vector thus produced with $(\cos\theta_1 + \mathrm{j}\sin\theta_1)$ will give a further rotation $\theta_1$, *i.e.*, a total rotation $(\theta_1 + \theta_2)$. But this is also the rotation which would be produced by $\cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2)$. Therefore,

$$(\cos\theta_1 + \mathrm{j}\sin\theta_1)(\cos\theta_2 + \mathrm{j}\sin\theta_2)\mathbf{r} = \{\cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2)\}\mathbf{r} \quad (43)$$

or, equating the operators,

$$(\cos\theta_1 + \mathrm{j}\sin\theta_1)(\cos\theta_2 + \mathrm{j}\sin\theta_2) = \cos(\theta_1 + \theta_2) + \mathrm{j}\sin(\theta_1 + \theta_2). \quad (44)$$

Expressing both these operators in the alternative form $\epsilon^{\mathrm{j}\theta}$ deduced in Par. 12,

$$\epsilon^{\mathrm{j}\theta_1} \cdot \epsilon^{\mathrm{j}\theta_2} = \epsilon^{\mathrm{j}(\theta_1+\theta_2)}, \quad (45)$$

which is in accordance with the index law.

[1] Bibliography, No. 24.

The above process can obviously be extended to show that

$$\epsilon^{j\theta_1}\cdot\epsilon^{j\theta_2}\cdot\epsilon^{j\theta_3}\ \ldots \text{ to } n \text{ terms} = \epsilon^{j(\theta_1+\theta_2+\theta_3+\ \ldots \text{ to } n \text{ terms})} \qquad (46)$$

and, putting $\theta_1 = \theta_2 = \theta_3 = \ \ldots \ = \theta_n = \theta$ (47)

then

$$(\epsilon^{j\theta})^n = \epsilon^{jn\theta}, \qquad (48)$$

where $n$ is any positive integer.

It should be noted that, in the above, no restrictions have been placed on the sign of $\theta$, which may be positive or negative.

Again

$$\left\{(\epsilon^{j\theta})^{\frac{1}{n}}\right\}^n = \epsilon^{j\theta} \qquad (49)$$

irrespective of the interpretation of $(\epsilon^{j\theta})^{\frac{1}{n}}$.

Thus, $(\epsilon^{j\theta})^{\frac{1}{n}}$ is that operation which, repeated $n$ times, produces a total rotation $\theta$, *i.e.*,

$$(\epsilon^{j\theta})^{\frac{1}{n}} = \epsilon^{\frac{j\theta}{n}}. \qquad (50)$$

Further, since

$$\epsilon^{j\theta}\cdot\frac{1}{\epsilon^{j\theta}} = 1, \qquad (51)$$

$\frac{1}{\epsilon^{j\theta}}$ is the operator which cancels the effect of $\epsilon^{j\theta}$, *i.e.*, $\frac{1}{\epsilon^{j\theta}}$ produces a rotation $-\theta$. Therefore,

$$\frac{1}{\epsilon^{j\theta}} = \epsilon^{-j\theta}. \qquad (52)$$

In virtue of the above (Eqs. (33), (35), (36)), it may be stated that, when the operator $(\cos\theta + j\sin\theta)$ is expressed in the equivalent form $\epsilon^{j\theta}$, the $j\theta$ has the properties of an ordinary index.

**14. The Sign and the Magnitude of $\theta$.**—In the discussion of the operator $(a + jb)$ no restrictions have been placed on the

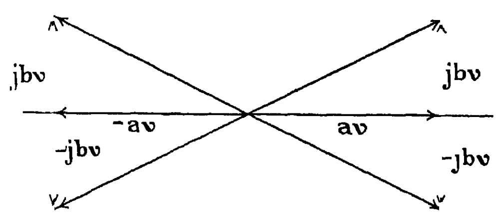


FIG. 13.

sign or the magnitude of a and b, which can be any real positive or negative numbers. It will now be well to consider the relationship which will exist between the magnitude of $\theta$ and the signs of a and b.

The relationship can easily be derived from a consideration of Fig. 13, and can be tabulated thus:

| a | b | $\theta$ lies between |
|---|---|---|
| + | + | 0° and 90° |
| − | + | 90° and 180° |
| − | − | 180° and 270° |
| + | − | 270° and 360° |

Reference to the above table should prevent any possibility of ambiguity in the sign and the magnitude of $\theta$ when operators of the form $(a + jb)$ are expressed in the equivalent form $r\epsilon^{j\theta}$.

**15. Some Important Properties of Vector Operators.**—1. If $(a + jb)\mathbf{v} = 0$, *i.e.*, if $(a + jb) = 0$, then $a = 0$ and $b = 0$. Otherwise since

$$a\mathbf{v} + jb\mathbf{v} = 0, \tag{53}$$

one would have,

$$a\mathbf{v} = -jb\mathbf{v}, \tag{54}$$

*i.e.*, a vector equal to another vector which is perpendicular to it, which, by the definition of a vector, is impossible. Another way of expressing the same thing is the obvious statement that a displacement in one direction can only be canceled by a displacement in an opposite direction.

2. If

$$(a_1 + jb_1)\mathbf{v} = (a_2 + jb_2)\mathbf{v}, \tag{55}$$

*i.e.*, if

$$(a_1 + jb_1) = (a_2 + jb_2), \tag{56}$$

then

$$a_1 = a_2 \text{ and } b_1 = b_2, \tag{57}$$

for

$$a_1\mathbf{v} + jb_1\mathbf{v} = a_2\mathbf{v} + jb_2\mathbf{v}, \tag{58}$$

*i.e.*,

$$a_1\mathbf{v} - a_2\mathbf{v} + jb_1\mathbf{v} - jb_2\mathbf{v} = 0 \tag{59}$$

or,

$$(a_1 - a_2)\mathbf{v} + j(b_1 - b_2)\mathbf{v} = 0. \tag{60}$$

Therefore, as shown above,

$$a_1 - a_2 = 0 \text{ and } b_1 - b_2 = 0 \tag{61}$$

or

$$a_1 = a_2 \text{ and } b_1 = b_2. \tag{62}$$

Remembering that $a = r\cos\theta$

and $b = r\sin\theta$,

it is easily seen that the above propositions take the alternative forms.

(*a*) If

$$r\epsilon^{j\theta} = 0 \tag{63}$$

then

$$r = 0 \text{ and } \theta \text{ is indeterminate.} \tag{64}$$

(b) If $$r_1\epsilon^{j\theta_1} = r_2\epsilon^{j\theta_2} \tag{65}$$
then $$r_1 = r_2 \text{ and } \theta_1 = \theta_2. \tag{66}$$

These two propositions, particularly in their first forms, will prove to be of considerable value in the application of the vectorial method to the solution of electrical problems, and should be carefully noted.

**16. Combinations of Operators.**—1. It was shown in Par. 7 that the result of the addition (or the subtraction) of any number of vectors is independent of the order in which the various steps are performed. It follows from this that

$$(a_1 + jb_1) + (a_2 + jb_2) + (a_3 + jb_3) + \text{etc., etc.} \tag{67}$$
$$= (a_1 + jb_1 + a_2 + jb_2 + a_3 + jb_3 + \text{etc. etc.})$$
$$= (a_1 + a_2 + a_3 + \text{etc.}) + (jb_1 + jb_2 + jb_3 + \text{etc.}) \tag{68}$$
$$(a_1 + jb_1) + (a_2 + jb_2) + (a_3 + jb_3) + \text{etc.} =$$
$$(a_1 + a_2 + a_3 + \text{etc.}) + j(b_1 + b_2 + b_3 + \text{etc.}). \tag{69}$$

The addition or the subtraction of operators, therefore, consists of the combination of the "a" and "b" parts separately so as to form a single expression of the type $\Sigma a + j\Sigma b$. It is to be understood in the above that the a's and b's are unrestricted in sign.

2. The application of vectorial methods to the solution of certain practical problems will sometimes give rise to more or less complicated functions of operators involving products, powers, quotients, roots, etc, of expressions such as $(a + jb)$.

The simple product of two such expressions does not involve any difficulty, for

$$(a + jb)(c + jd) = a(c + jd) + jb(c + jd) \tag{70}$$
$$= ac + ajd + jbc + jbjd \tag{71}$$
$$= ac + jad + jbc + j^2bd \tag{72}$$
$$= ac + jad + jbc - bd \tag{73}$$
$$= (ac - bd) + j(ad + bc). \tag{74}$$

The product can, therefore, be reduced to a single expression of the form $(a + jb)$ by applying the ordinary rules for multiplication, together with the fact that $j \times j = -1$.

In the more complicated expressions, however, such as

$$\frac{(a_1 + jb_1)^2\sqrt{(a_2 + jb_2)}}{(a_3 + jb_3)^5},$$

a much more convenient and more general method of reduction to the simple standard forms is to express each of the component operators in the form $r\epsilon^{j\theta}$. Thus, for the example given above

$$\frac{(a_1 + jb_1)^2 \sqrt{a_2 + jb_2}}{(a_3 + jb_3)^5} = \frac{(r_1\epsilon^{j\theta_1})^2 \sqrt{r_2\epsilon^{j\theta_2}}}{(r_3\epsilon^{j\theta_3})^5}. \tag{75}$$

Since, as shown in Par. 13, the indices in the above expression obey the index laws, then

$$\frac{(a_1 + jb_1)^2 \sqrt{a_2 + jb_2}}{(a_3 + jb_3)^5} = \frac{r_1^2 r_2^{\frac{1}{2}}}{r_3^5} \epsilon^{j(2\theta_1 + \frac{\theta_2}{2} - 5\theta_3)}. \tag{76}$$

Thus, the operator function has been reduced to the form $r\epsilon^{j\theta}$, where

$$r = r_1^2 r_2^{\frac{1}{2}} r_3^{-5} \tag{77}$$

and

$$\theta = \left(2\theta_1 + \frac{\theta_2}{2} - 5\theta_3\right). \tag{78}$$

It can now, if desired, be put in the form $(a + jb)$, since

$$a = r\cos\theta \text{ and } b = r\sin\theta. \tag{79}$$

To take a numerical example,

$$\frac{(4 + j5)^2 \sqrt{7 + j9}}{(8 + j)^5} =$$

$$\frac{(\sqrt{41})^2 \sqrt[4]{130}}{65^{\frac{5}{2}}} \epsilon^{j(2 \tan^{-1} 1.25 + \frac{1}{2} \tan^{-1} 1.29 - 5 \tan^{-1} .125)}$$

$$= 4.064 \times 10^{-3} \epsilon^{j(2 \times 51°21' + \frac{1}{2} 52°12' - 5 \times 7°8')}$$

$$= 4.064 \times 10^{-3} \epsilon^{j\,93°8'}$$

$$= 4.064 \times 10^{-3}(-\cos 3°\,8' + j \sin 3°\,8')$$

$$= -3.98 \times 10^{-3} + j\,2.22 \times 10^{-4}.$$

**17. Some Special Cases.**—1. A case which frequently occurs in practice is the inverse operator to $(a + jb)$, *i.e.*, $\frac{1}{(a + jb)}$.

Applying the method of the previous paragraph,

$$\frac{1}{a + jb} = \frac{1}{r\epsilon^{j\theta}} = \frac{1}{r}\epsilon^{-j\theta}, \tag{80}$$

whence it is clear that $\frac{1}{a + jb}$ divides the magnitude of its operand by r and rotates it through an angle $\theta$ in a negative, *i.e.*, clockwise, sense.

In a simple form such as $\frac{1}{a + jb}$ it may be just as convenient to operate top and bottom with what may be called the "conjugate operator," in this case $(a - jb)$, *i.e.*,

$$\frac{1}{a + jb} = \frac{a - jb}{(a + jb)(a - jb)} \tag{81}$$

$$= \frac{a - jb}{a^2 - (jb)^2} \tag{82}$$

$$= \frac{a - jb}{a^2 + b^2} \tag{83}$$

$$= \frac{a}{a^2 + b^2} - j\frac{b}{a^2 + b^2}. \tag{84}$$

2. Similarly,
$$\frac{a_1 + jb_1}{a_2 + jb_2} = \frac{r_1}{r_2}\epsilon^{j(\theta - \theta_2)} \tag{85}$$

or alternatively,
$$\frac{a_1 + jb_1}{a_2 + jb_2} = \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \tag{86}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}. \tag{87}$$

3. Applying the above methods to $\frac{1}{j}$,

$$\frac{1}{j} = \frac{1}{j}\cdot\frac{j}{j} = -j \tag{88}$$

or
$$\frac{1}{j} = \frac{1}{\epsilon^{j\tan^{-1}\infty}} = \epsilon^{-j\tan^{-1}\infty}$$

$$= \epsilon^{-j\frac{\pi}{2}}, \tag{89}$$

*i.e.*, $\frac{1}{j}$ produces a rotation of $\frac{\pi}{2}$ in a negative (*i.e.*, clockwise) sense, which is also obvious from first principles.

4. Similary,
$$1 + j = \sqrt{2}\,\epsilon^{j\frac{\pi}{4}} \tag{90}$$

$$1 - j = \sqrt{2}\,\epsilon^{-j\frac{\pi}{4}} \tag{91}$$

$$\frac{1}{1 + j} = \frac{1}{\sqrt{2}}\epsilon^{-j\frac{\pi}{4}}. \tag{92}$$

5. For $\sqrt{j}$,
$$\sqrt{j} = \sqrt{\epsilon^{j\frac{\pi}{2}}} \tag{93}$$

$$= \epsilon^{j\frac{\pi}{4}}. \tag{94}$$

Putting this in the form $(a + jb)$,

$$a = r\cos\theta = \cos 45° = \frac{1}{\sqrt{2}} \tag{95}$$

$$b = r\sin\theta = \sin 45° = \frac{1}{\sqrt{2}} \tag{96}$$

$$\therefore\ \sqrt{j} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 + j). \tag{97}$$

**18. Trigonometrical and Exponential Functions of Vector Operators.**—Paragraphs 15 and 16 indicate how any expression involving sums, differences, products, powers, etc. of terms such

as (a + jb) can be reduced to either of the standard forms for purposes of calculation. In the more elementary applications of the vectorial method, only expressions of this comparatively simple character are likely to arise. Cases will occur, however, as in the vectorial solution of problems relating to distributed capacities and inductances, in which operator functions of a different character will appear, *e.g.*, sin (a + jb), sinh (a + jb) $\epsilon^{PX}$, where P is itself a vector operator, etc.

The reduction of such expressions to the standard form will not in general present any greater difficulty than those already considered. The discussion of such cases will, however, be deferred until the necessity arises.

**19. Graphic Calculations with Vectors and Operators.**—One of the chief advantages of the vectorial method is its graphic character and the readiness with which it lends itself to the graphic solution of problems to which it is applied. Below are a few examples of the graphic representation of some of the operations considered in the preceding paragraphs.

1. *The Graphic Representation of Operators.*—If it is desired to determine graphically the values of operator combinations, the

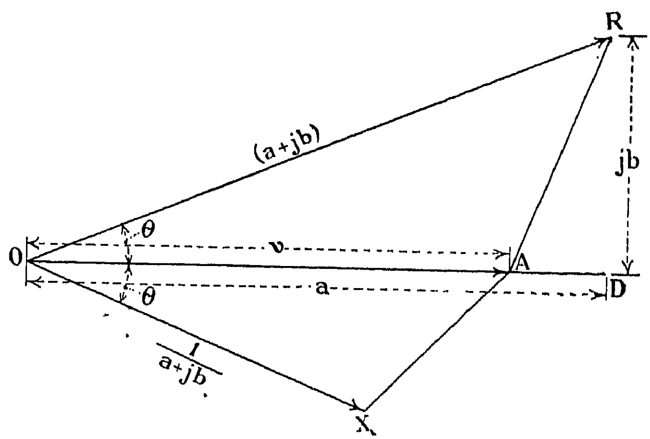


FIG. 14.

component operators (a + jb), etc. can be represented by the vectors (a + jb)v, etc. Thus, referring to Fig. 14, the line OR can be considered to represent the operator (a + jb), or $r\epsilon^{j\theta}$.[1]

[1] Where the vector (a + jb)v is used to represent the operator (a + jb), the symbol v can conveniently be omitted.

2. *The Determination of* $\frac{1}{(a + jb)}$.—Referring to Fig. 14, if the triangle OXA be drawn so as to be similar to the triangle OAR, then OX is the vector $\frac{\mathbf{v}}{(a + jb)}$, so that if OR represents the operator (a + jb), then OX will represent the operator $\frac{1}{(a + jb)}$.

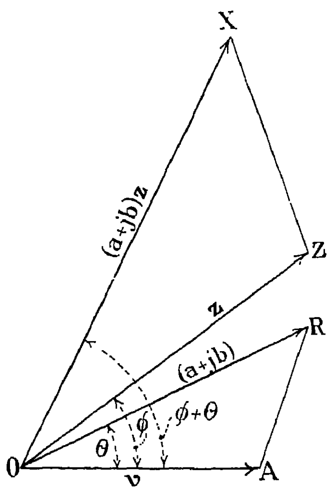


FIG. 15.

This follows from the fact that

$$\frac{OX}{OA} = \frac{OA}{OR}, \tag{98}$$

*i.e.*,

$$\frac{OX}{1} = \frac{1}{r}. \tag{99}$$

Further, the angle $A\hat{O}X$ is $-\theta$.

The drawing of the triangle OXA is a matter of elementary practical geometry and need not be described in detail.

3. *The Graphic Representation of* (a + jb)**z**.—Referring to Fig. 15, if OZ represents the vector **z**, of magnitude z, and OR represents the operator (a + jb)—*i.e.*, OR is the vector (a + jb)**v**—then, drawing the triangle OZX similar to the triangle OAR, OX will represent the vector (a + jb)**z**, for

$$\frac{OX}{OZ} = \frac{OR}{OA} \tag{100}$$

*i.e.*,

$$\frac{OX}{z} = \frac{r}{1} \tag{101}$$

or

$$OX = rz. \tag{102}$$

Also, the angle $X\hat{O}Z$ is equal to the angle $R\hat{O}A$, *i.e.*,

$$X\hat{O}Z = \theta \tag{103}$$

and

$$X\hat{O}A = (\phi + \theta). \tag{104}$$

Similarly, if the vector **z** be (c + jd)**v**, then OX will be the vector (a + jb)(c + jd)**v**, *i.e.*, the line OX can be considered to represent the operator (a + jb)(c + jd).

As a special case of this, if

$$(a + jb) = (c + jd), \tag{105}$$

then OX will represent the operator $(a + jb)^2$. This is illustrated in Fig. 16, which is the same as Fig. 15 when Z coincides with R.

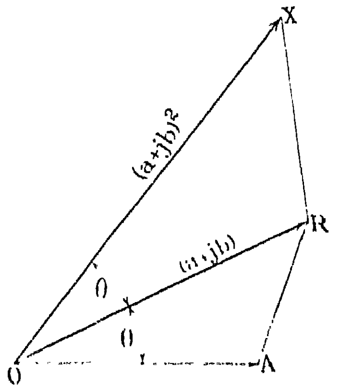


FIG. 16.

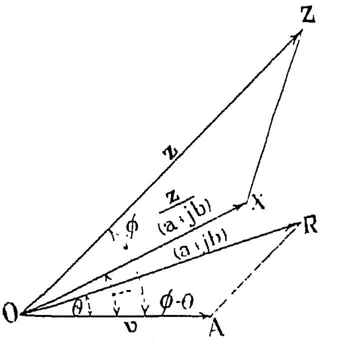


FIG. 17.

4. *The Graphic Representation of* $\frac{z}{(a + jb)}$.—Referring to Fig. 17, if OZ represents the vector **z** and OR represents the operator $(a + jb)$, then, drawing the triangle OZX similar to the triangle ORA, OX will represent the vector $\frac{z}{(a + jb)}$, for

$$\frac{OX}{OZ} = \frac{OA}{OR} \tag{106}$$

*i.e.*,

$$\frac{OX}{z} = \frac{1}{r} \tag{107}$$

or

$$OX = \frac{z}{r}. \tag{108}$$

Also

$$ZOX = \theta. \tag{109}$$

Similarly, if OZ be the operator $(c + jd)$, then OX will represent the operator $\frac{(c + jd)}{(a + jb)}$.

The above and similar constructions, of which it is unnecessary to give further illustrations, will be of use in the application of vectorial methods to problems relating to alternating currents.

## EXAMPLES

1. Draw the unit vector **v**, the vector 6**v**, and the vector 5j**v**. Draw the four vectors $\pm 6v \pm j5v$ and write each of them in the form $r\epsilon^{j\theta}$.
2. Express the operators $10\epsilon^{\pm j25°}$ in the form $a + jb$.
3. Express the operator $\sqrt[4]{(7 + j11)}$ in the form $r\epsilon^{j\theta}$.

**4.** State the limits of $\theta$ in the operators $(\pm a \pm jb)$ and $(\pm a \pm jb)^2$, a and b being positive numbers.

**5.** Express in the forms $(a + jb)$ and $r\epsilon^{j\theta}$ the operators:

(*a*) $j^{\frac{1}{n}}$.

(*b*) $j^{\frac{p}{q}}$.

(*c*) $(a + jb)^{\frac{p}{q}}$.

(*d*) $\dfrac{(2 + j3)^{\frac{1}{1}}\,(4 + j5)}{(2 + j3)^{\frac{1}{2}}\,(6 + j7)^{\frac{3}{2}}}$

**6.** Given that $Z = 4\epsilon^{j30^\circ}$ v,
Determine graphically:

(*a*) $(2 + j2.5)Z$.

(*b*) $\dfrac{1}{(2 + j2.5)}\,Z$.

(*c*) $\sqrt{(2 + j2.5)}\;Z$.

(*d*) $\dfrac{1}{\sqrt{(2 + j2.5)}}\,Z$.

Write the results in each case in the form $r\epsilon^{j\theta}$ v.

## ANSWERS TO EXAMPLES

**1.** $6v + j5v;\ r = 7.8;\ \theta = 39^\circ\,48'$.
$6v - j5v;\ r = 7.8;\ \theta = -39^\circ\,48'$.
$-6v + j5v;\ r = 7.8;\ \theta = 140^\circ\,12'$.
$-6v - j5v;\ r = 7.8;\ \theta = -140^\circ\,12'$.

**2.** $10\epsilon^{\pm j25^\circ} = 9.06 \pm j4.23$.

**3.** $1.9\epsilon^{j14^\circ 23'}$.

**4.** $a + jb;\ 0^\circ - 90^\circ$.
$-a + jb;\ 90^\circ - 180^\circ$.
$-a - jb;\ 180^\circ - 270^\circ$.
$a - jb;\ 270^\circ - 360^\circ$.
$a^2 - b^2 + 2abj;\ a > b;\ \ 0^\circ - 90^\circ$
$a < b;\ 90^\circ - 180^\circ$
$a^2 - b^2 - 2abj;\ a < b;\ 180^\circ - 270^\circ$
$a > b;\ 270^\circ - 360^\circ$

**5.** (*a*) $\cos\dfrac{90^\circ}{n} + j\sin\dfrac{90^\circ}{n};\ \epsilon^{\frac{j90^\circ}{n}}$

(*b*) $\cos\dfrac{90p^\circ}{q} + j\sin\dfrac{90p^\circ}{q};\ \epsilon^{\frac{j90p^\circ}{q}}$

(*c*) $r^{\frac{p}{q}}\cos\dfrac{p\theta}{q} + j\sin\dfrac{p\theta}{q};\ r^{\frac{p}{q}}\,\epsilon^{\frac{jp\theta}{q}}$

where $r^2 = a^2 + b^2$ and $\tan\theta = \dfrac{b}{a}$.

(*d*) $.1328 - j.0996;\ .166\epsilon^{-j36^\circ\,50'}$

**6.** (*a*) $12.81\,\epsilon^{j81^\circ 21'}$ v.
(*b*) $1.25\,\epsilon^{-j21^\circ 21'}$ v.
(*c*) $7.157\,\epsilon^{j55^\circ 40'}$ v.
(*d*) $2.24\,\epsilon^{j4^\circ 20'}$ v.

# CHAPTER II

## THE SCALAR PRODUCT OF VECTORS

**20. Definition of the Scalar Product of Two Vectors.**[1]—*The scalar product of two vectors is the product of their magnitudes multiplied by the cosine of the angle between their positive directions.*

In the case illustrated in Fig. 18 the scalar product of $\mathbf{r}_1$ and $\mathbf{r}_2$, which will be written $\mathbf{r}_1 \cdot \mathbf{r}_2$, is $r_1 r_2 \cos\theta$, $r_1$ and $r_2$ being the magnitudes of $\mathbf{r}_1$ and $\mathbf{r}_2$.

It is important to realize that a scalar product, as its name indicates, is a scalar quantity, *i.e.*, a number. Thus, if $\mathbf{r}_1$, is of magnitude 5, $\mathbf{r}_2$ of magnitude 10, and $\theta$ is 60°, then

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos\theta = 5 \times 10 \times \tfrac{1}{2} = 25. \tag{110}$$

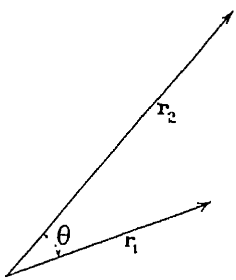


FIG. 18.

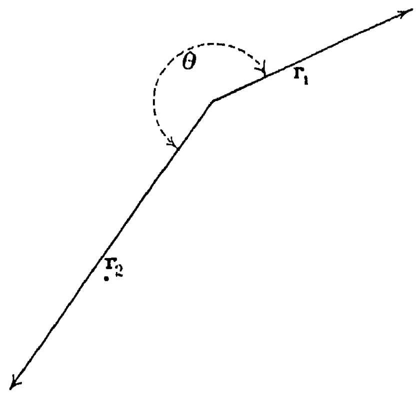


FIG. 19.

It is equally important to realize, however, that a scalar or, as it is sometimes called, a "dot" product, like any other scalar quantity, can have sign. In the applications of the conception of scalar product, sign has usually an important physical significance.

Suppose, for instance, that $\mathbf{r}_1$ represents the displacement of a body under the action of a system of forces, one of which is represented in magnitude and direction by $\mathbf{r}_2$. Then the scalar product $\mathbf{r}_1 \cdot \mathbf{r}_2$ represents the amount of work done by the force

[1] Bibliography, No. 22.

on the body. Now let $\mathbf{r}_2$ be reversed in direction (Fig. 19). Then $\theta$ becomes 240° instead of 60°, and $\cos\theta$ is $-\frac{1}{2}$ instead of $\frac{1}{2}$, so that $\mathbf{r}_1 \cdot \mathbf{r}_2$ is $-25$. In this case the work done by the force on the body is negative. In other words, work is done *by* the body *against* the force.

The physical significance of the sign of the scalar product will appear more fully in the applications of the vectorial method to the solution of problems relating to alternating currents.

**21. Some Important Special Cases of Scalar Product.**—Taking as the general form

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathrm{r}_1\mathrm{r}_2 \cos\theta. \tag{111}$$

1. If

$$\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}, \tag{112}$$

then

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r} \cdot \mathbf{r} \tag{113}$$

$$= \mathrm{r}^2 \cos 0 \tag{114}$$

$$= \mathrm{r}^2, \tag{115}$$

*i.e., the scalar square of a vector is the square of its scalar magnitude.* Provided its real meaning be clearly understood, the scalar square of $\mathbf{r}$ can be written $\mathbf{r}^2$. Thus, in general,

$$\mathbf{r}^2 = \mathrm{r}^2. \tag{116}$$

In particular,

$$\mathbf{v}^2 = 1 \tag{117}$$

$$\mathrm{j}\mathbf{v}^2 = 1 \tag{118}$$

(It should be noted that $\mathrm{j}\mathbf{v}^2$ can only mean $\mathrm{j}\mathbf{v} \cdot \mathrm{j}\mathbf{v}$. $\mathrm{j}(\mathbf{v}^2)$ is a meaningless expression, since j requires a vector operand, and $\mathbf{v}^2$ is not a vector.)

2. If $\mathbf{r}_1$ is perpendicular to $\mathbf{r}_2$,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1\mathbf{r}_2 \cos\frac{\pi}{2} \tag{119}$$

$$= 0. \tag{120}$$

Thus, the scalar product of any two mutually perpendicular vectors is zero.

In particular,

$$\mathrm{j}\mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot \mathrm{j}\mathbf{v} = 0. \tag{121}$$

**22. The Scalar Multiplication of Vectors.**—Referring to Fig. 20:

$$\mathbf{r}_1 \cdot \mathbf{r}_4 = \mathrm{OA} \cdot \mathrm{OC} \cos\theta_4 \tag{122}$$

$$= \mathrm{OA} \cdot \mathrm{OE} \tag{123}$$

$$= \mathrm{OA}(\mathrm{OD} + \mathrm{DE}) \tag{124}$$

$$= \mathrm{OA} \cdot \mathrm{OD} + \mathrm{OA} \cdot \mathrm{DE} \tag{125}$$

$$= \mathrm{OA} \cdot \mathrm{OB} \cos\theta_2 + \mathrm{OA} \cdot \mathrm{BC} \cos\theta_3 \tag{126}$$

$$= \mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_3. \tag{127}$$

But

$$\mathbf{r}_4 = \mathbf{r}_2 + \mathbf{r}_3 \tag{128}$$

$$\therefore\ \mathbf{r}_1 \cdot (\mathbf{r}_2 + \mathbf{r}_3) = \mathbf{r}_1 \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_3. \tag{129}$$

Further, suppose that $\mathbf{r}_1$ is itself the sum of two vectors $\mathbf{r}_5$ and $\mathbf{r}_6$, *i.e.*,

$$\mathbf{r}_1 = \mathbf{r}_5 + \mathbf{r}_6. \tag{130}$$

Substituting this in Eq. (129),

$$(\mathbf{r}_5 + \mathbf{r}_6) \cdot (\mathbf{r}_2 + \mathbf{r}_3) = (\mathbf{r}_5 + \mathbf{r}_6) \cdot \mathbf{r}_2 + (\mathbf{r}_5 + \mathbf{r}_6) \cdot \mathbf{r}_3 \tag{131}$$

$$= \mathbf{r}_5 \cdot \mathbf{r}_2 + \mathbf{r}_6 \cdot \mathbf{r}_2 + \mathbf{r}_5 \cdot \mathbf{r}_3 + \mathbf{r}_6 \cdot \mathbf{r}_3 \tag{132}$$

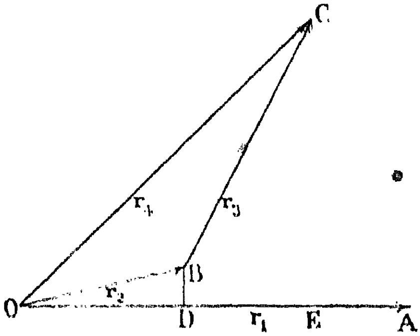


FIG. 20.

The method can be extended to the scalar product of combinations of any number of vectors. The result can be summarized thus:

*The scalar multiplication of vectors follows the ordinary algebraic law of distribution.*

**23. Scalar Products and Vector Operators.** 1. Since $(a + jb)\mathbf{v}$ is really the sum of the vectors $a\mathbf{v}$ and $jb\mathbf{v}$ then, from the preceding paragraph,

$$(a + jb)\mathbf{v} \cdot \mathbf{v} = a\mathbf{v} \cdot \mathbf{v} + jb\mathbf{v} \cdot \mathbf{v} \tag{133}$$

$$= a\mathbf{v} \cdot \mathbf{v} + bj\mathbf{v} \cdot \mathbf{v} \tag{134}$$

$$= a. \tag{135}$$

Similarly,

$$(a + jb)\mathbf{v} \cdot j\mathbf{v} = bj\mathbf{v} \cdot j\mathbf{v} \tag{136}$$

$$= b. \tag{137}$$

Thus, any vector $\mathbf{r}$ in the plane of the paper can be expressed directly in the form

$$\mathbf{r} = \{(\mathbf{r} \cdot \mathbf{v}) + j(\mathbf{r} \cdot j\mathbf{v})\}\mathbf{v}. \tag{138}$$

This is obviously a vectorial form of statement for Eq. (24).

2. If $\mathbf{r}_1 = (a_1 + jb_1)\mathbf{v}$ (139)

and $\mathbf{r}_2 = (a_2 + jb_2)\mathbf{v}$, (140)

then

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = (a_1a_2)\mathbf{v} \cdot \mathbf{v} + (b_1b_2)j\mathbf{v} \cdot j\mathbf{v} + (a_1b_2)\mathbf{v} \cdot j\mathbf{v} + (b_1 \cdot a_2)j\mathbf{v} \cdot \mathbf{v} \tag{141}$$

$$= (a_1a_2 + b_1b_2).$$

It should be noted at this point that $(a_1 + jb_1)\mathbf{v} \cdot (a_2 +$ . is a different quantity from $(a_1 + jb_1)\ (a_2 + jb_2)\mathbf{v}$.

The former, as shown, is the scalar quantity $(a_1a_2 + b$ The latter is a vector which, in terms of $a_1, a_2, b_1, b_2$, can be wri $\{(a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1)\}\mathbf{v}$.

Alternatively, in the r and $\theta$ form

$$(a_1 + jb_1)\mathbf{v} \cdot (a_2 + jb_2)\mathbf{v} = \mathbf{r}_1 \cdot \mathbf{r}_2 \qquad (1$$

$$= r_1r_2 \cos(\theta_1 - \theta_2) \qquad (1$$

and $$(a_1 + jb_1)(a_2 + jb)\mathbf{v} = r_1r_2\, \epsilon^{j(\theta_1+\theta_2)}\mathbf{v}. \qquad (1$$

The distinction is emphasized since $(a_1 + jb_1)(a_2 + jb_2)$ sometimes wrongly described as the product of two vect instead of as the product of two operators (or, alternative as successive operations).

3. *j and Scalar Products.*—Let OA and OB (Fig. 21) represe two vectors $\mathbf{r}_1$, $\mathbf{r}_2$ of magnitudes $r_1$ and $r_2$.

Then $$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1r_2 \cos \theta. \qquad (14$$

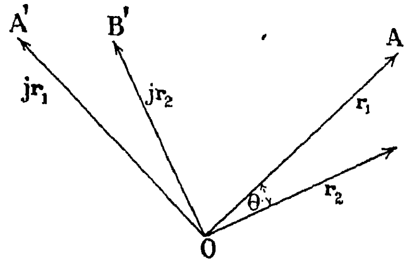


FIG. 21.

Let OA′ perpendicular to $\mathbf{r}_1$ and OB′ perpendicular to $\mathbf{r}$ represent $j\mathbf{r}_1$ and $j\mathbf{r}_2$ respectively.

Then $$j\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1r_2 \cos \text{BOA}' \qquad (147)$$

$$= r_1r_2 \cos\left(\frac{\pi}{2} + \theta\right) \qquad (148)$$

$$= -r_1r_2 \sin \theta. \qquad (149)$$

On the other hand,

$$\mathbf{r}_1 \cdot j\mathbf{r}_2 = r_1r_2 \cos\left(\frac{\pi}{2} - \theta\right) \qquad (150)$$

$$= r_1r_2 \sin \theta. \qquad (151)$$

Therefore, $$j\mathbf{r}_1 \cdot \mathbf{r}_2 = -\mathbf{r}_1 \cdot j\mathbf{r}_2. \qquad (152)$$

*Thus, transferring the operator* j *from one member of a scalar product to the other does not alter the magnitude of the product, but it reverses its sign.* Further, if it is desired to transfer an operator

of the form (a + jb) from one member of a scalar product to the other, then

$$\mathbf{r}_1 \cdot (a + jb)\mathbf{r}_2 = \mathbf{r}_1 \cdot a\mathbf{r}_2 + \mathbf{r}_1 \cdot jb\mathbf{r}_2 \qquad (153)$$
$$= a\mathbf{r}_1 \cdot \mathbf{r}_2 + b\mathbf{r}_1 \cdot j\mathbf{r}_2 \qquad (154)$$
$$= a\mathbf{r}_1 \cdot \mathbf{r}_2 - jb\mathbf{r}_1 \cdot \mathbf{r}_2 \qquad (155)$$
$$= (a - jb)\mathbf{r}_1 \cdot \mathbf{r}_2. \qquad (156)$$

Expressed in the r, $\theta$ form, this becomes

$$\mathbf{r}_1 \cdot r\epsilon^{j\theta}\mathbf{r}_2 = r\epsilon^{-j\theta}\mathbf{r}_1 \cdot \mathbf{r}_2. \qquad (157)$$

4. $(a + jb)\mathbf{r}_1 \cdot \mathbf{v}$ in Terms of $\mathbf{r}_1 \cdot \mathbf{v}$.—Given that

$$\mathbf{r}_1 \cdot \mathbf{v} = r_1 \cos \theta_1, \qquad (158)$$

then
$$(a + jb)\mathbf{r}_1 \cdot \mathbf{v} = a\mathbf{r}_1 \cdot \mathbf{v} + jb\,\mathbf{r}_1 \cdot \mathbf{v} \qquad (159)$$
$$= ar_1 \cos \theta_1 + br_1 \cos \left(\theta_1 + \frac{\pi}{2}\right) \qquad (160)$$
$$= ar_1 \cos \theta_1 - br_1 \sin \theta_1. \qquad (161)$$

Alternatively, if $\mathbf{r}_1 \cdot \mathbf{v} = r_1 \cos \theta_1, \qquad (162)$

then
$$r\epsilon^{j\theta}\mathbf{r}_1 \cdot \mathbf{v} = rr_1 \cos (\theta_1 + \theta), \qquad (163)$$

since the effect of the operator $r\epsilon^{j\theta}$ is to rotate its operand through an angle $\theta$ and to multiply the magnitude of its operand by r. Thus, if

$$r\epsilon^{j\theta} = (a + jb) \qquad (164)$$

and

$$\mathbf{r}_1 \cdot \mathbf{v} = r_1 \cos \theta_1, \qquad (165)$$

then $r\epsilon^{j\theta}\,\mathbf{r}_1 \cdot \mathbf{v}$ can be expressed in either of the forms

$$rr_1 \cos (\theta_1 + \theta) \qquad (166)$$

or

$$ar_1 \cos \theta_1 - br_1 \sin \theta_1. \qquad (167)$$

The identity of the two forms is obvious, since

$$a = r \cos \theta \qquad (168)$$

and

$$b = r \sin \theta. \qquad (169)$$

This general result is of great importance in the applications of vectorial notation to problems involving alternating currents.

**EXAMPLES**

**1.** Evaluate the following scalar products:

(*a*) $(21\epsilon^{j7°}\mathbf{v}) \cdot (21\epsilon^{j7°}\mathbf{v})$.
(*b*) $(21\epsilon^{j7°}\mathbf{v}) \cdot (21\epsilon^{j67°}\mathbf{v})$.
(*c*) $(j21\epsilon^{j7°}\mathbf{v}) \cdot (21\epsilon^{j67°}\mathbf{v})$.
(*d*) $(10\epsilon^{j18°}\mathbf{v}) \cdot (20\epsilon^{j63°}\mathbf{v})$.
(*e*) $j^{\frac{1}{3}}(3 + j4)\mathbf{v} \cdot (5 + j6)\mathbf{v}$.
(*f*) $(2\epsilon^{j60°} + 3\epsilon^{j30°})\mathbf{v} \cdot (5\epsilon^{j10°})\mathbf{v}$.
(*g*) $(3 \cdot 598 + j3 \cdot 232)\mathbf{v} \cdot (5\epsilon^{j10°})\mathbf{v}$.
(*h*) $(113 + j209)\mathbf{v} \cdot (-209 + j113)\mathbf{v}$.

2. Express as the sum of a number of scalar products

$$(\mathbf{a} + \mathbf{b} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{e} - \mathbf{f}).$$

3. What is the magnitude of $(a + jb)\mathbf{v} \cdot (1 + j)\mathbf{v}$?

4. If

$$\mathbf{r} \cdot \mathbf{v} = \text{const.} = k$$

and one end of $\mathbf{r}$ is fixed in position, what is the locus of the other end?

5. If

$$(\mathbf{r} - \mathbf{a})^2 = \text{const.} = k^2$$

and one end of $\mathbf{r}$ is fixed in position, what is the locus of the other end? $\mathbf{a}$ being a constant vector.

6. Given that

$$\mathbf{r}_1 \cdot \mathbf{v} = \mathbf{r}_1 \cos \theta_1,$$

Express $\left\{ \dfrac{\mathbf{r}_1}{(a + jb)} \right\} \cdot \mathbf{v}$ in terms of $r_1$, $\theta_1$, a, and b, and in terms of $r_1$, r, $\theta_1$, and $\theta$, where

$$r^2 = a^2 + b^2$$

and

$$\tan \theta = \frac{b}{a}.$$

## ANSWERS TO EXAMPLES

1. (*a*) 441.
   (*b*) 220·5.
   (*c*) 365.
   (*d*) 141·4.
   (*e*) 32·8.
   (*f*) 20·53.
   (*g*) 20·53.
   (*h*) 0.
2. $\mathbf{a} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{e} - \mathbf{a} \cdot \mathbf{f} + \mathbf{b} \cdot \mathbf{d} - \mathbf{b} \cdot \mathbf{e} - \mathbf{b} \cdot \mathbf{f} - \mathbf{c} \cdot \mathbf{d} + \mathbf{c} \cdot \mathbf{e} + \mathbf{c} \cdot \mathbf{f}.$
3. $(a + b)$.
4. A straight line perpendicular to $\mathbf{v}$ and distant k from the fixed end of $\mathbf{r}$.
5. A circle of radius k, the center of which is distant a in the direction $\mathbf{a}$ from the fixed end of $\mathbf{r}$.
6. $\dfrac{(ar_1 \cos \theta_1 + br_1 \sin \theta_1)}{\sqrt{a^2 + b^2}}$; $\dfrac{r_1}{r} \cos (\theta_1 - \theta)$.

# CHAPTER III

## VARIABLE VECTORS

**24. Vector Functions of Time.**—Up to this point vectors have been regarded as constant quantities. Certain vectors must now be considered of which the magnitude or the direction, or both, are functions of some independent variable, such as time.

For the specification of such vectors the scalar product with some fixed unit vector, such as $\mathbf{v}$, is a convenient form, since it defines both the magnitude and the slope of the vector.

Consider, for instance, a vector defined by

$$\mathbf{r} \cdot \mathbf{v} = \mathrm{r} \cos \theta \qquad (170)$$

$$\theta = \text{const.} \qquad (171)$$

$$\mathrm{r} = \mathrm{r}_0 + \mathrm{kt}. \qquad (172)$$

This is clearly a vector of constant direction whose magnitude is proportional to time, being $\mathrm{r}_0$ at the instant $\mathrm{t} = 0$.

Again, the vector defined by

$$\mathbf{r} \cdot \mathbf{v} = \mathrm{r} \cos \theta \qquad (173)$$

$$\mathrm{r} = \mathrm{r}_0 = \text{const.} \qquad (174)$$

$$\theta = \omega \mathrm{t} + \psi \ (\omega \text{ and } \psi \text{ being consts.}) \qquad (175)$$

is a vector of constant magnitude $\mathrm{r}_0$, rotating with constant angular velocity $\omega$.

As a further example, consider the case in which

$$\mathbf{r} \cdot \mathbf{v} = \mathrm{r} \cos \theta \qquad (176)$$

$$\mathrm{r} = \mathrm{r}_0 \epsilon^{-\mathrm{kt}} \quad (\mathrm{r}_0 \text{ and k being consts.}) \qquad (177)$$

$$\theta = (\omega \mathrm{t} + \psi) \quad (\omega \text{ and } \psi \text{ being consts.}), \qquad (178)$$

*i.e.*,

$$\mathbf{r} \cdot \mathbf{v} = \mathrm{r}_0 \epsilon^{-\mathrm{kt}} \cos (\omega \mathrm{t} + \psi) \qquad (179)$$

This is a vector of exponentially decreasing magnitude and constant angular velocity. The coefficient k in the above is known as the damping constant.

**25. The Differentiation of Vectors.**[1]—Consider a vector $\mathbf{r}$ whose magnitude and position relative to the fixed unit vector $\mathbf{v}$ are dependent on the independent variable time. At the instant

[1] Bibliography, No. 22.

t let it be represented by the line OR, and at the instant $t + \delta t$ by the line OR′ (Fig. 22). Let OR′ be the vector $\mathbf{r} + \delta\mathbf{r}$. Then RR′ is the vector $\delta\mathbf{r}$. In accordance with the usual notation, *the limiting value of* $\frac{\delta \mathbf{r}}{\delta t}$ *when* t *tends to zero will be written* $\frac{d\mathbf{r}}{dt}$ *and will be termed the "differential coefficient" of the vector* $\mathbf{r}$ *with respect to* t.

It must be noted that $\frac{d\mathbf{r}}{dt}$ is itself a vector, whose direction with respect to $\mathbf{v}$ will depend on the nature of the dependence of $\mathbf{r}$ on t. Thus, if $\mathbf{r}$ is constant in direction, $\delta\mathbf{r}$ is parallel to $\mathbf{r}$ and

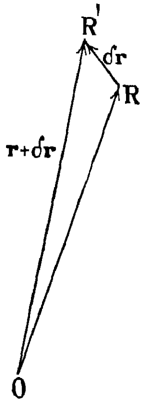


FIG. 22.

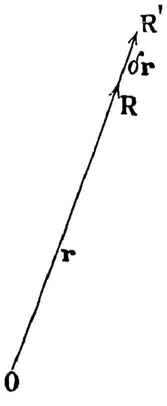


FIG. 23.

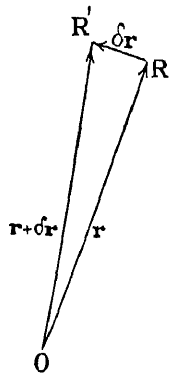


FIG. 24.

$\frac{d\mathbf{r}}{dt}$ will also be parallel to $\mathbf{r}$ (Fig. 23). On the other hand, if $\mathbf{r}$ is constant in magnitude, then when $\delta\mathbf{r}$ becomes vanishingly small it will be perpendicular to $\mathbf{r}$, and $\frac{d\mathbf{r}}{dt}$ will therefore be perpendicular to $\mathbf{r}$ (Fig. 24).

For the complete specification of $\frac{d\mathbf{r}}{dt}$ it is, therefore, necessary to know how both the magnitude and the direction of $\mathbf{r}$ vary with time.

**26. General Expression for** $\frac{d\mathbf{r}}{dt}$.—As before, let OR be $\mathbf{r}$ and OR′, $\mathbf{r} + \delta\mathbf{r}$. Let RP be perpendicular to OR′ (Fig. 25).

In the limit, when $\delta\mathbf{r}$ tends to zero,

$$\mathrm{RP} = r\delta\theta \tag{180}$$

and is perpendicular to $\mathbf{r}$. As a vector, therefore, RP can be written $r\delta\theta j\frac{\mathbf{r}}{r}$, since $j\frac{\mathbf{r}}{r}$ is the unit vector perpendicular to $\mathbf{r}$ in the direction RP.

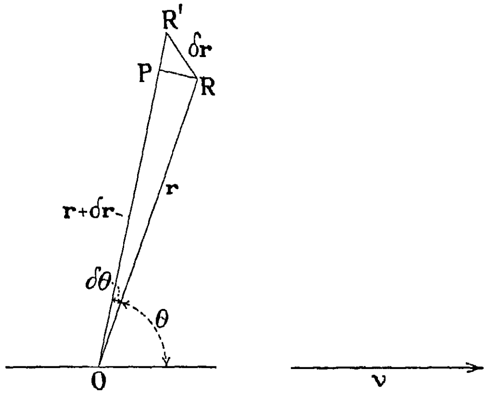


Fig. 25.

Also, when $\delta\theta$ tends to zero, PR′, the magnitude of which is $\delta r$, is parallel to $\mathbf{r}$ and can be written as a vector $\delta r\frac{\mathbf{r}}{r}$, since $\frac{\mathbf{r}}{r}$ is the unit vector parallel to $\mathbf{r}$.

$$\therefore \quad \delta\mathbf{r} = \delta r\frac{\mathbf{r}}{r} + r\delta\theta j\frac{\mathbf{r}}{r} \tag{181}$$

$$\therefore \quad \frac{\delta\mathbf{r}}{\delta t} = \frac{\delta r}{\delta t}\frac{\mathbf{r}}{r} + \frac{r\delta\theta}{\delta t} j\frac{\mathbf{r}}{r} \tag{182}$$

and in the limit
$$\frac{d\mathbf{r}}{dt} = \left(\frac{1}{r}\frac{dr}{dt} + j\frac{d\theta}{dt}\right)\mathbf{r}. \tag{183}$$

Thus, in general,
$$\frac{d\mathbf{r}}{dt} = (a + jb)\mathbf{r} \tag{184}$$

where
$$a = \frac{1}{r}\frac{dr}{dt} \tag{185}$$

and
$$b = \frac{d\theta}{dt}. \tag{186}$$

Special interest attaches to those cases in which a and b are constant with respect to time. In all such cases

$$\frac{d\mathbf{r}}{dt} = (a + jb)\mathbf{r} \tag{187}$$

$$\frac{d}{dt}\frac{d\mathbf{r}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = (a + jb)\frac{d\mathbf{r}}{dt}, \tag{188}$$

$$= (a + jb)^2\mathbf{r}. \tag{189}$$

Similarly, by continuing the process,

$$\frac{d^n r}{dt^n} = (a + jb)^n \mathbf{r}. \tag{190}$$

**27. Vectors of Constant Magnitude Rotating with Constant Angular Velocity.**—It is clear that vectors of this type, defined by Eqs. (173) to (175), satisfy the above conditions relating to a and b, since

$$b = \frac{d\theta}{dt} = \omega = \text{const.} \tag{191}$$

$$a = \frac{1}{r}\frac{dr}{dt} = 0. \tag{192}$$

Thus, for any vector such that

$$\mathbf{r}\cdot\boldsymbol{\nu} = r_0 \cos(\omega t + \psi) \tag{193}$$

$$\frac{d\mathbf{r}}{dt} = \omega j \mathbf{r} \tag{194}$$

and, in general,

$$\frac{d^n \mathbf{r}}{dt^n} = (\omega j)^n \mathbf{r}. \tag{195}$$

**28. Vectors of Exponentially Decreasing Magnitude, Rotating with Constant Angular Velocity.**—Any vector for which

$$\mathbf{r}\cdot\boldsymbol{\nu} = r_0\,\epsilon^{-kt} \cos(\omega t + \psi) \tag{196}$$

will also satisfy the same conditions, for, since $r = r_0\epsilon^{-kt}$,

$$a = \frac{1}{r}\frac{dr}{dt} = -k = \text{const.} \tag{197}$$

and

$$b = \frac{d\theta}{dt} = \omega = \text{const.}, \tag{198}$$

so that

$$\frac{d\mathbf{r}}{dt} = (-k + \omega j)\mathbf{r} \tag{199}$$

and

$$\frac{d^n \mathbf{r}}{dt^n} = (-k + \omega j)^n \mathbf{r}. \tag{200}$$

This result will prove of value in the application of vectorial methods to the solution of problems relating to exponentially damped oscillations.

**29. The Differentiation of Scalar Products.**—The only case which need be considered for present purposes is that in which one member of the scalar product is constant.

Let

$$\mathbf{r}\cdot\mathbf{r}_0 = rr_0 \cos\theta \quad (\mathbf{r}_0 \text{ being const.}). \tag{201}$$

Then

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}_0) = \frac{d}{dt}(rr_0\cos\theta) \tag{202}$$

$$= r_0 \cos\theta \frac{\partial r}{\partial t} \mp rr_0 \sin\theta \frac{d\theta}{dt}$$

$$= \mathrm{rr}_0 \cos \theta \cdot \frac{1}{\mathrm{r}} \frac{\mathrm{dr}}{\mathrm{dt}} + \mathrm{rr}_0 \cos \left(\theta + \frac{\pi}{2}\right) \frac{d\theta}{\mathrm{dt}} \tag{203}$$

$$= (\mathbf{r} \cdot \mathbf{r}_0) \frac{1}{\mathrm{r}} \frac{\mathrm{dr}}{\mathrm{dt}} + (\mathrm{j}\mathbf{r} \cdot \mathbf{r}_0) \frac{d\theta}{\mathrm{dt}} \tag{204}$$

$$= \left(\frac{1}{\mathrm{r}} \frac{\mathrm{dr}}{\mathrm{dt}} + \mathrm{j} \frac{d\theta}{\mathrm{dt}}\right) \mathbf{r} \cdot \mathbf{r}_0 \tag{205}$$

$$= \frac{d\mathbf{r}}{\mathrm{dt}} \cdot \mathbf{r}_0. \tag{206}$$

The result is thus analogous to the corresponding case in scalar differentiation.

An important special case is that in which $\mathbf{r}_0 = \mathbf{v}$, when

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{r} \cdot \mathbf{v}) = \frac{d\mathbf{r}}{\mathrm{dt}} \cdot \mathbf{v}. \tag{207}$$

**30. Equations Involving Scalar Products of Variable Vectors.**—1. Let $\mathbf{r}$ be a vector rotating with constant angular velocity $\omega$.

If

$$\mathbf{r} \cdot \mathbf{v} = 0 \tag{208}$$

for all values of t, then either

$$\mathbf{r} = 0 \tag{209}$$

or $\mathbf{r}$ is perpendicular to $\mathbf{v}$ (Par. 21).

But, since $\mathbf{r}$ rotates with constant angular velocity, it cannot be perpendicular to $\mathbf{v}$ for all values of t,

$$\therefore \quad \mathbf{r} = 0. \tag{210}$$

2. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two vectors rotating with constant angular velocity $\omega$.

Then, if

$$\mathbf{r}_1 \cdot \mathbf{v} = \mathbf{r}_2 \cdot \mathbf{v} \tag{211}$$

for all values of t,

$$\mathbf{r}_1 = \mathbf{r}_2. \tag{212}$$

This follows immediately from Eq. (208), as can be seen by putting

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{213}$$

The above propositions are fundamental in the application of the vectorial method, since, as will appear later, they provide a means of transforming equations involving simple harmonic functions of time and their differential coefficients into equations in rotating vectors and vector operators.

**31. Vectors Related by a Constant Operator.**—The equation

$$\mathbf{r}_2 = (\mathrm{a} + \mathrm{jb})\mathbf{r}_1, \tag{214}$$

where a and b are constants with respect to time, implies that

$$\frac{\mathrm{r}_2}{\mathrm{r}_1} = (\mathrm{a}^2 + \mathrm{b}^2)^{\frac{1}{2}} = \text{const.} \tag{215}$$

and that the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ is $\tan^{-1}\frac{b}{a}$, which is also constant. It follows, therefore, that whatever be the nature of the variation of $r_1$ with time, $r_2$ will vary in the same way. Thus, if $r_1$ is constant, $r_2$ is also constant. If $r_1$ varies according to an exponential law

$$r_1 = r_{10}\epsilon^{-kt}, \tag{216}$$

then the variation of $r_2$ with time can also be expressed

$$r_2 = r_{20}\epsilon^{-kt}. \tag{217}$$

It also follows that at every instant the two vectors must have the same angular velocity, since the angle between them remains constant for all values of t.

Thus, if between any two rotating vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ a relationship of the form $\mathbf{r}_2 = (a + jb)\mathbf{r}_1$ can be found which is true for all values of t, then the two vectors must have the same angular velocity and their magnitudes must vary in a similar manner with respect to t.

**32. The Product of Two Scalar Products.**—Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two rotating vectors of constant and equal angular velocity $\omega$ and of magnitudes $r_1$ and $r_2$.

Let
$$\mathbf{r}_1 \cdot \mathbf{v} = r_1 \cos \theta_1 \tag{218}$$
$$= r_1 \cos \omega t \tag{219}$$
and
$$\mathbf{r}_2 \cdot \mathbf{v} = r_2 \cos \theta_2 \tag{220}$$
$$= r_2 \cos (\omega t + \psi). \tag{221}$$
Then
$$(\mathbf{r}_1 \cdot \mathbf{v})(\mathbf{r}_2 \cdot \mathbf{v}) = r_1 r_2 \cos \theta_1 \cos \theta_2 \tag{222}$$
$$= \frac{r_1 r_2}{2}\{\cos (\theta_1 - \theta_2) + \cos (\theta_1 + \theta_2)\} \tag{223}$$
$$= \frac{r_1 r_2}{2} \cos \psi + \frac{r_1 r_2}{2} \cos (2\omega t + \psi). \tag{224}$$

The first term can, if desired, be expressed in the form $\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{2}$ and the second in the form $\mathbf{w} \cdot \mathbf{v}$, where $\mathbf{w}$ is a vector of magnitude $\frac{r_1 r_2}{2}$ which makes with $\mathbf{v}$ an angle equal to the sum of the angles made by $\mathbf{r}_1$ and $\mathbf{r}_2$ with $\mathbf{v}$, *i.e.*, which rotates with twice the angular velocity of $\mathbf{r}_1$ or $\mathbf{r}_2$.

Thus,
$$(\mathbf{r}_1 \cdot \mathbf{v})(\mathbf{r}_2 \cdot \mathbf{v}) = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{2} + \mathbf{w} \cdot \mathbf{v}. \tag{225}$$

This equation has an important bearing on the determination of energy conditions in certain electrical problems.

## EXAMPLES

**1.** If $\mathbf{r}\cdot\mathbf{v} = \mathrm{r}\cos\theta$
where $\mathrm{r} = \mathrm{r}_0\epsilon^{\mathrm{kt}}$
and $\theta = (\omega t + \psi)$ ($\mathrm{r}_0$, k, $\omega$, and $\psi$ being consts.),
give the operational expressions for $\dfrac{d\mathbf{r}}{dt}$ and $\dfrac{d^n\mathbf{r}}{dt^n}$.

**2.** If $\mathbf{r}\cdot\mathbf{v} = \mathrm{r}\cos\theta$
where $\mathrm{r} = \mathrm{r}_0\epsilon^{+\mathrm{kt}}$
and $\theta = (\omega t + \psi)$ ($\mathrm{r}_0$, k, $\omega$, and $\psi$ being consts.),
give expressions for $\left(\dfrac{d\mathbf{r}}{dt}\right)\cdot\mathbf{v}$ and $\left(\dfrac{d^2\mathbf{r}}{dt^2}\right)\cdot\mathbf{v}$.

Also show that
$$\frac{d(\mathbf{r}\cdot\mathbf{v})}{dt} = \left(\frac{d\mathbf{r}}{dt}\right)\cdot\mathbf{v}.$$

**3.** (*a*) Given that $\mathbf{r}$ is a vector of constant magnitude r rotating with constant angular velocity $\omega$, and that
$$a\left(\frac{d^2\mathbf{r}}{dt^2}\right) + b\left(\frac{d\mathbf{r}}{dt}\right) + c\mathbf{r} = \mathbf{k},$$
express $\mathbf{r}$ in terms of $\mathbf{k}$ and an operator.
(*b*) What kind of vector is $\mathbf{k}$?
(*c*) If k = 0, what equations will hold for a, b, and c?

**4.** Given that $\mathbf{E} = (\mathrm{R} + j\mathrm{X})\mathbf{I}$,
where $\mathbf{E}$ is a vector of constant magnitude $\hat{\mathrm{E}}$ rotating with constant angular velocity $\omega$, and R and X are consts., and that
$$e = \mathbf{E}\cdot\mathbf{v} = \hat{\mathrm{E}}\cos\omega t$$
$$i = \mathbf{I}\cdot\mathbf{v},$$
write the expressions for: (*a*) the instantaneous value of ie, (*b*) the mean value of ie during one revolution of $\mathbf{E}$, in terms of:

(1) $\mathbf{I}$ and $\mathbf{E}$.
(2) $\hat{\mathrm{I}}$ and $\hat{\mathrm{E}}$.
(3) $\hat{\mathrm{E}}$, R, and X.
(4) $\hat{\mathrm{I}}$, R, and X.

## ANSWERS TO EXAMPLES

**1.** $(k + \omega j)\mathbf{r}$, $(k + \omega j)^n\mathbf{r}$.

**2.** $\sqrt{\omega^2 + k^2}\,\mathrm{r}_0\epsilon^{-kt}\cos\left(\omega t + \psi + \tan^{-1} - \dfrac{\omega}{k}\right)$
$$(k^2 + \omega^2)\mathrm{r}_0\epsilon^{-kt}\cos\left\{\omega t + \psi + \tan^{-1} - \frac{2k\omega}{(k^2 - \omega^2)}\right\}.$$

**3.** (*a*)
$$\mathbf{r} = \frac{\mathbf{k}}{(c - a\omega^2) + b\omega j}.$$
(*b*) k is a vector of constant magnitude
$$(\sqrt{(c - a\omega^2)^2 + b^2\omega^2})\mathrm{r}$$
rotating with constant angular velocity $\omega$.

(*c*)
$$b = 0 \text{ and } \omega^2 = \frac{c}{a}.$$

**4.** (*a*)
$$(1)\ \left(\frac{\mathbf{E}\cdot\mathbf{I}}{2} + \mathbf{W}\cdot\mathbf{v}\right),$$

where **W** is a vector of magnitude $\frac{\hat{I}\hat{E}}{2}$ rotating with constant angular velocity $2\omega$.

(2) $\frac{\hat{E}\hat{I}}{2}\cos\theta + \frac{\hat{E}\hat{I}}{2}\cos(2\omega t - \theta)$

where $$\tan\theta = \frac{X}{R}.$$

(3) $\frac{\hat{E}^2 R}{2(R^2 + X^2)} + \frac{\hat{E}^2}{2\sqrt{R^2 + X^2}}\cos\left(2\omega t - \tan^{-1}\frac{X}{R}\right).$

(4) $\frac{\hat{I}^2 R}{2} + \frac{\hat{I}^2}{2}\sqrt{R^2 + X^2}\cos\left(2\omega t - \tan^{-1}\frac{X}{R}\right).$

(b) (1) $\frac{\mathbf{E}\cdot\mathbf{I}}{2}.$

(2) $\frac{\hat{E}\hat{I}}{2}\cos\theta.$

(3) $\frac{\hat{E}^2 R}{2(R^2 + X^2)}.$

(4) $\frac{\hat{I}^2 R}{2}.$

## CHAPTER IV

## THE APPLICATION OF VECTOR ANALYSIS TO ALTERNATING CURRENTS

**33. The Representation of an Alternating Current as a Scalar Product.**—Let I be a vector of constant magnitude $\hat{I}$ rotating in a positive (*i.e.*, counterclockwise) sense with constant angular velocity $\omega$, its slope relative to $\mathbf{v}$ being $\psi$ at the instant $t = 0$. If i be its scalar product with $\mathbf{v}$, then

$$i = \mathbf{I} \cdot \mathbf{v} \tag{226}$$
$$= \hat{I} \cos \theta \tag{227}$$
$$= \hat{I} \cos (\omega t + \psi). \tag{228}$$

*Thus, an alternating current of amplitude* $\hat{I}$ *and frequency* $\frac{\omega}{2\pi}$ *can be represented as the scalar product with* $\mathbf{v}$ *of a vector* I *of constant magnitude* $\hat{I}$ *rotating with uniform angular velocity* $\omega$.

It is this fact which underlies the vector diagrams of alternating current theory, and which makes it possible to apply to alternating currents the principles enunciated in the preceding chapters. The form of statement usually found in textbooks is that the instantaneous value of the alternating current is represented by the projection in a given fixed direction of the rotating vector. In the opinion of the author, the use of vectorial notation and of the term "scalar product" is preferable to the usual form of statement, since it enables the relation between the alternating current and the vector which is used to represent it to be expressed in the form of an exact equation

$$i = \mathbf{I} \cdot \mathbf{v}. \tag{229}$$

**The "Effective" or Root-mean-square Value of an Alternating Current or Potential Difference.**—For reasons which will appear more fully later, the quantity actually used in practice to specify the magnitude of an alternating current or potential difference is that known as the "root-mean-square value" of the quantity. It is the square root of the mean value during one period of the square of the instantaneous magnitude. Thus, if the instantane-

ous magnitude of an alternating current of period T be i, the root-mean-square value, denoted by I, is given by

$$I = \left\{ \frac{1}{T}\int_0^T i^2 dt \right\}^{\frac{1}{2}}. \tag{230}$$

The relation of this root-mean-square value to the amplitude of the alternating current can be simply demonstrated by the proposition established in Par. 32. Putting, as above,

$$i = \mathbf{I} \cdot \mathbf{v}, \tag{231}$$

then

$$i^2 = (\mathbf{I} \cdot \mathbf{v})(\mathbf{I} \cdot \mathbf{v}) \tag{232}$$

$$= \frac{\mathbf{I} \cdot \mathbf{I}}{2} + \mathbf{W} \cdot \mathbf{v}, \tag{233}$$

where $\mathbf{W}$ is a vector of magnitude $\frac{\hat{I}^2}{2}$ rotating with twice the angular velocity of the vector $\mathbf{I}$. The quantity $\mathbf{W} \cdot \mathbf{v}$ is, therefore, a purely periodic function, the mean value of which during the period T is zero. Further,

$$\frac{\mathbf{I} \cdot \mathbf{I}}{2} = \frac{\hat{I}^2}{2} = \text{const.} \tag{234}$$

The square root of the mean value of the right-hand side of Eq. (233) is, therefore, given by

$$\left(\frac{1}{T}\int_0^T i^2 dt\right)^{\frac{1}{2}} = \sqrt{\frac{\hat{I}^2}{2}} = \frac{\hat{I}}{\sqrt{2}}. \tag{235}$$

This is a general result of great practical importance, which can be expressed as follows:

*The root-mean-square value of a pure sine-wave alternating current or potential difference is equal to the amplitude or maximum value of the current or potential difference divided by the square root of* 2.

In the actual construction of vector diagrams it will usually be more convenient to make the lengths of all the vectors concerned proportional to the root-mean-square values of the electrical quantities represented by them. This will not, of course, in any way affect the theory developed in this and subsequent chapters, since it is merely equivalent to an alternation of the linear scale on which the vectors are drawn.

**34. The Vectorial Expression of Kirchhoff's Laws.**—1. Kirchhoff's first law can be expressed in the following way:

*The algebraic sum of the currents which meet at any point in a f conductors is zero.*

It must be noted that the law refers to the algebraic sum, *i.e.*, to the sum having regard to sign. By "sign" is here meant the direction of the currents with respect to the point considered. The usual convention in this matter, and the one which will be adhered to in this work, is that a current will be reckoned positive if it is flowing towards the point and negative if it is flowing away from the point. Thus, in Fig. 26,

$$i_1 + i_2 + i_3 - i_4 = 0, \tag{236}$$

and in Fig. 27

$$i_1 + i_2 + i_3 + i_4 = 0 \tag{237}$$

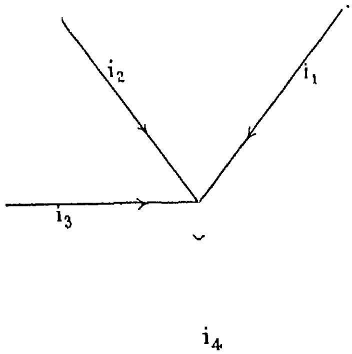


FIG. 26.

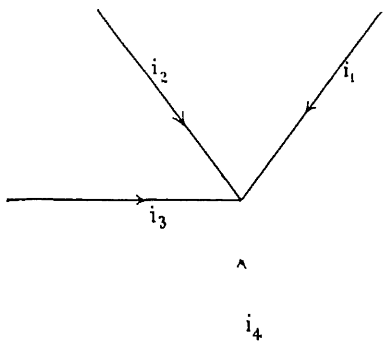


FIG. 27.

If the currents are alternating currents of pure sine-wave form, then, as shown in Par. 33, i can be expressed as $\mathbf{I} \cdot \mathbf{v}$, $i_2$ as $\mathbf{I}_2 \cdot \mathbf{v}$, etc. giving

$$\mathbf{I}_1 \cdot \mathbf{v} + \mathbf{I}_2 \cdot \mathbf{v} + \mathbf{I}_3 \cdot \mathbf{v} + \mathbf{I}_4 \cdot \mathbf{v} = 0 \tag{238}$$

*i.e.*,

$$(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4) \cdot \mathbf{v} = 0 \tag{239}$$

and, since this is true at every instant, it follows from Par. 30 that

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 = 0. \tag{240}$$

The vectorial form of statement for Kirchhoff's first law, therefore, is expressed:

*In any network of conductors carrying alternating currents of sine-wave form, the vector sum of the vectors representing the currents which meet at any point is zero.*

2. Kirchhoff's second law may be expressed:

*In any closed circuit in a network of conductors the algebraic sum of the potential differences is zero.*

As in the first law, it is the algebraic sum which is considered. The usual convention with regard to the sign of a potential difference, and the one which will be adhered to in this work, is that a certain direction round a circuit is considered positive, and potential differences which tend to drive a current in this positive direction are considered positive, and *vice versa.* Thus, referring to Fig. 28, which represents a source of continuous potential sending a current through a resistance, if $e_R$ be the back e.m.f. or potential difference across the ends of the resistance,

$$e + e_R = 0. \tag{241}$$

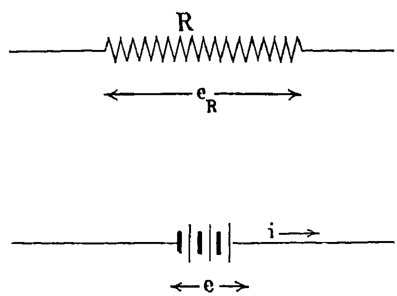


Fig. 28.

By Ohm's law, the magnitude of $e_R$ is iR, and its direction is opposite to that of the current, so that

$$e_R = -iR. \tag{242}$$

Therefore,
$$e - iR = 0 \tag{243}$$

and
$$e = iR \tag{244}$$

or
$$i = \frac{e}{R}. \tag{245}$$

By a method exactly similar to that employed in the first law, and which, therefore, need not be stated in detail, it can be shown that Kirchhoff's second law can be applied to the vectors representing the alternating potential differences in any closed circuit in a network of conductors carrying alternating currents. The law may, therefore, be expressed:

*In any closed circuit in a network of conductors carrying alternating currents of sine-wave form the vector sum of the vectors representing the potential differences is zero.*

**35. Current and Potential Relationships in Vectorial Form.**—Before the above two laws can be applied to particular cases it will be necessary to obtain general expressions for the relationships between the vectors representing the potential differences and the current for conductors of various types, *e.g.*, a resistance, an inductance, and a capacity.

1. *Resistance.*—Referring to Fig. 28, let the source of continuous potential be replaced by an alternating e.m.f., as shown in Fig. 29, and let i, the instantaneous value of the current produced

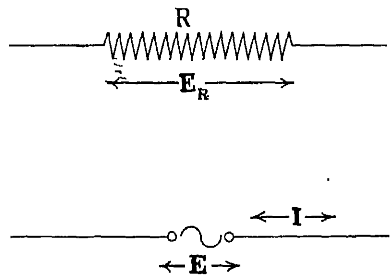


FIG. 29.

by it, be represented by $\mathbf{I} \cdot \mathbf{v}$. For $e_R$, the instantaneous value of the potential difference across the resistance,

$$e_R = -iR. \tag{246}$$

Representing $e_R$ by $\mathbf{E}_R \cdot \mathbf{v}$,

$$\mathbf{E}_R \cdot \mathbf{v} = -R(\mathbf{I} \cdot \mathbf{v}) \tag{247}$$

$$= -R\mathbf{I} \cdot \mathbf{v} \tag{248}$$

and, since this is true for all values of t,

$$\mathbf{E}_R = -R\mathbf{I}. \tag{249}$$

Also, by Kirchhoff's second law

$$\mathbf{E} + \mathbf{E}_R = 0. \tag{250}$$

Therefore, $$\mathbf{E} = -\mathbf{E}_R = R\mathbf{I}. \tag{251}$$

Thus, the vector representing i is a constant multiple of the vector representing e. *The vector* **I** *is, therefore in the same*

*direction as* **E**, *rotating in the same sense with the same angular velocity, and is of magnitude* $\frac{1}{R}$ *times that of* **E**. This is illustrated in Fig. 30.

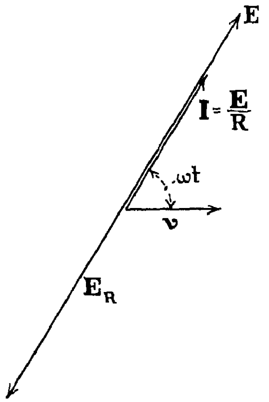


Fig. 30.

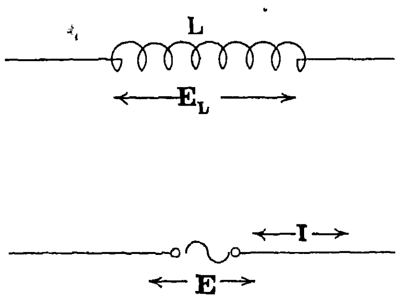


Fig. 31.

2. *Inductance.*—Referring to Fig. 31, which represents a pure inductance L in series with a source of alternating e.m.f., if $e_L$ be the instantaneous value of the potential difference across L

$$e_L = -L\frac{di}{dt}. \qquad (252)$$

Therefore,

$$\mathbf{E}_L \cdot \mathbf{v} = -L\frac{d(\mathbf{I} \cdot \mathbf{v})}{dt} \qquad (253)$$

$$= -L\left(\frac{d\mathbf{I}}{dt}\right) \cdot \mathbf{v} \qquad (254)$$

and, since this is true for all values of t,

$$\mathbf{E}_L = -L\frac{d\mathbf{I}}{dt}. \qquad (255)$$

Also, since

$$\mathbf{E} + \mathbf{E}_L = 0 \qquad (256)$$

$$\mathbf{E} = -\mathbf{E}_L = L\frac{d\mathbf{I}}{dt}. \qquad (257)$$

Thus, $\frac{d\mathbf{I}}{dt}$ is a vector of constant magnitude $\frac{\hat{E}}{L}$ rotating with constant angular velocity $\omega$. Therefore, **I** must also be a vector of constant magnitude rotating with constant angular velocity $\omega$, in virtue of Par. 27 and, therefore,

$$\frac{dI}{dt} = \omega j I, \tag{258}$$

so that

$$\mathbf{E} = \omega j L I \tag{259}$$

or

$$I = \frac{\mathbf{E}}{\omega j L}. \tag{260}$$

*Thus, the vector representing* i *is one of constant magnitude* $\frac{\hat{E}}{\omega L}$ *and constant angular velocity* $\omega$, *making with* **E** *an angle* $-90°$, *as shown in Fig.* 32.

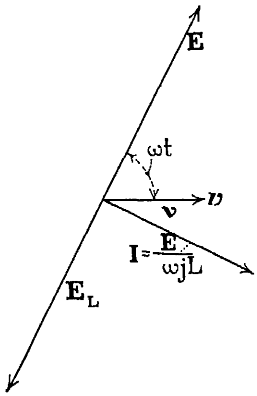


FIG. 32.

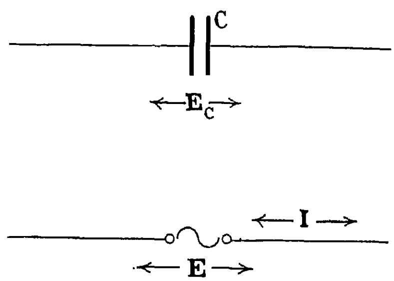


FIG. 33.

3. *Capacity.*—For the case illustrated in Fig 33, if $e_c$ be the instantaneous value of the potential difference between the plates of the condenser C, and if q be the instantaneous value of the charge on the plates, then

$$e_c = -\frac{q}{C} \tag{261}$$

and

$$\frac{de_c}{dt} = -\frac{1}{C}\frac{dq}{dt} \tag{262}$$

$$= -\frac{i}{C}, \tag{263}$$

since

$$i = \frac{dq}{dt}. \tag{264}$$

Therefore,

$$\frac{d(\mathbf{E}_c \cdot \mathbf{v})}{dt} = -\frac{(\mathbf{I} \cdot \mathbf{v})}{C}. \tag{265}$$

Also, by Kirchhoff's second law,

$$\mathbf{E} + \mathbf{E}_c = 0, \tag{266}$$

so that $\mathbf{E}_c$ is a vector of constant angular velocity and constant magnitude. Therefore,

$$\frac{d(\mathbf{E}_c \cdot \mathbf{v})}{dt} = \left(\frac{d\mathbf{E}_c}{dt}\right) \cdot \mathbf{v} \tag{267}$$

$$= \omega j \mathbf{E}_c \cdot \mathbf{v}. \tag{268}$$

Therefore,

$$\omega j \mathbf{E}_c \cdot \mathbf{v} = -\left(\frac{I}{C}\right) \cdot \mathbf{v} \tag{269}$$

and, since this is true for all values of t,

$$\omega j \mathbf{E}_c = -\frac{I}{C} \tag{270}$$

and

$$\mathbf{I} = -\omega j C \mathbf{E}_c = \omega C j \mathbf{E}. \tag{271}$$

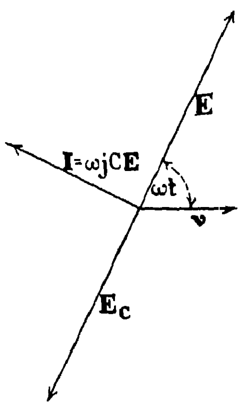


FIG. 34.

*Thus, the vector representing* i *is one of constant magnitude* $\omega C\hat{E}$, *making with* **E** *an angle* $+90°$ as shown in Fig. 34.

Therefore, for the three cases of a pure resistance, a pure inductance, and a pure capacity

$$\mathbf{E}_R = -RI \tag{272}$$

$$\mathbf{E}_L = -\omega j L I \tag{273}$$

$$\mathbf{E}_c = -\frac{I}{\omega j C}. \tag{274}$$

The above expressions are fundamental in the application of the vectorial method to problems involving alternating-current networks. In practice, it will not, of course, be necessary to repeat the intermediate steps which are required for a rigid demonstration of their validity. The currents and the potentials concerned can be expressed directly in vector form and the relationships between them as vector operators. In this vectorial form the analysis gains considerably in clearness and in simplicity, and is, moreover, convertible at will into either the usual scalar form or a vector diagram.

In the following paragraphs a number of illustrations of the method will be given. These are not, of course, intended to be a complete treatment of the subjects considered. They are simply intended as typical examples of the handling of the vectorial method.

**36. Resistance, Inductance, and Capacity in Series.**—For the circuit illustrated in Fig. 35, the application of Kirchhoff's second law in its vectorial form will give

$$\mathbf{E} + \mathbf{E}_R + \mathbf{E}_L + \mathbf{E}_C = 0; \tag{275}$$

therefore,

$$\mathbf{E} = -(\mathbf{E}_R + \mathbf{E}_L + \mathbf{E}_C). \tag{276}$$

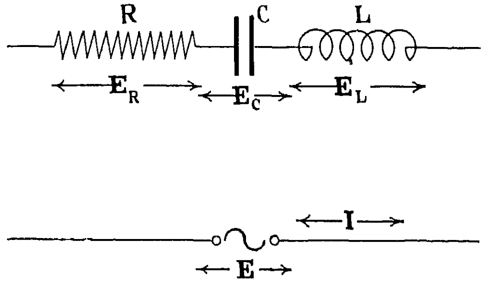


FIG. 35.

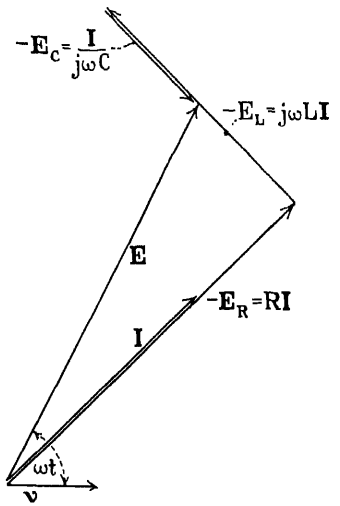


FIG. 36.

Substituting for $\mathbf{E}_R$, $\mathbf{E}_L$, and $\mathbf{E}_C$ their values in terms of $\mathbf{I}$,

$$\mathbf{E} = \left(\mathrm{R} + \omega j\mathrm{L} + \frac{1}{j\omega\mathrm{C}}\right)\mathrm{I} \tag{277}$$

$$= \left\{\mathrm{R} + j\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)\right\}\mathbf{I}. \tag{278}$$

Putting

$$\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}} = \mathrm{X}, \tag{279}$$

then

$$\mathbf{E} = (\mathrm{R} + j\mathrm{X})\mathbf{I} \tag{280}$$

or

$$\mathbf{I} = \frac{\mathbf{E}}{(\mathrm{R} + j\mathrm{X})}. \tag{281}$$

The vector diagram of the circuit will, therefore, be as shown in Fig. 36. The current vector is the result of operating on the

potential vector with (R + jX). This operator is usually described as the "vector impedance." The term is somewhat misleading, as (R + jX) is not a vector but an operator, and the name "impedance operator" is therefore greatly to be preferred. This term serves to distinguish (R + jX) from the expression $(R^2 + X^2)^{\frac{1}{2}}$, which is described as the "impedance." The angle $\tan^{-1}\frac{X}{R}$ is known as the "phase angle." It is, of course, the angle between the current and e.m.f. vectors, since

$$(R + jX) = (R^2 + X^2)^{\frac{1}{2}}\epsilon^{\,j\tan^{-1}\frac{X}{R}}, \tag{282}$$

*i.e.*,

$$\frac{1}{(R + jX)} = \frac{1}{(R^2 + X^2)^{\frac{1}{2}}}\epsilon^{-j\tan^{-1}\frac{X}{R}}. \tag{283}$$

If it is desired to express the above results in scalar form, then

$$\mathbf{I} = \frac{\mathbf{E}}{R + jX} \tag{284}$$

$$= \left(\frac{R}{R^2 + X^2} - \frac{jX}{R^2 + X^2}\right)\mathbf{E}. \tag{285}$$

Taking the scalar product of both sides of the equation with $\mathbf{v}$,

$$\mathbf{I}\cdot\mathbf{v} = \left(\frac{R}{R^2 + X^2} - \frac{jX}{R^2 + X^2}\right)\mathbf{E}\cdot\mathbf{v}. \tag{286}$$

But $\mathbf{I}\cdot\mathbf{v}$ is i, the instantaneous value of the alternating current. Therefore, since

$$\mathbf{E}\cdot\mathbf{v} = \hat{E}\cos\omega t \tag{287}$$

$$i = \frac{R\hat{E}}{R^2 + X^2}\cos\omega t - \frac{X\hat{E}}{R^2 + X^2}\cos(\omega t + 90^\circ) \tag{288}$$

$$= \frac{R\hat{E}}{R^2 + X^2}\cos\omega t + \frac{X\hat{E}}{R^2 + X^2}\sin\omega t. \tag{289}$$

Alternatively, putting

$$(R + jX) = z = z\epsilon^{j\phi} \tag{290}$$

where

$$z^2 = R^2 + X^2 \tag{291}$$

and

$$\phi = \tan^{-1}\frac{X}{R}, \tag{292}$$

then

$$i = \frac{1}{z}\mathbf{E}\cdot\mathbf{v} = \frac{1}{z}\epsilon^{-j\phi}\mathbf{E}\cdot\mathbf{v} \tag{293}$$

and, since the effect of the operator is to multiply its operand by $\frac{1}{z}$ and to rotate it through an angle $-\phi$,

$$i = \frac{\hat{E}}{z}\cos(\omega t - \phi). \tag{294}$$

**37. The General Impedance Operator.**[1]—It was shown in Par. 16 that any combination of sums, differences, products, quotients, etc. of operators of the type $(a + jb)$ could be replaced by a single operator of the same type. It follows from this that $(R + jX)$ is the most general type of impedance operator, and that any combination of series or parallel arrangements of conductors which are pure resistances, capacities, or inductances will oppose to an alternating e.m.f. of sine-wave form an impedance of this type. This does not mean that the R term in such a general impedance will consist of pure resistance terms only, or the X term of pure inductance or capacity terms only, but, however it may be constituted, the R term will be termed the "resistance component" of the impedance, and the X term the "reactance component."

**38. The General Admittance Operator.**—Putting

$$(R + jX) = z = z\epsilon^{j\phi}, \tag{295}$$

the expression $y$, defined by

$$yz = 1, \tag{296}$$

is termed the "general admittance operator."

Putting $y$ in the form

$$y = \frac{1}{z} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2} \tag{297}$$

$$= \frac{R}{z^2} - j\frac{X}{z^2}, \tag{298}$$

then $\frac{R}{z^2}$ is termed the "conductance component" of the admittance, and will be represented by the symbol K; and $\frac{X}{z^2}$ is described as the "susceptance component" of the admittance, and will be represented by the symbol S, *i.e.*,

$$y = K - jS. \tag{299}$$

Alternatively, $\quad y = y\epsilon^{-j\phi} \quad (300)$

where $\quad y = \frac{1}{z} \quad (301)$

and $\quad \phi = \tan^{-1}\frac{S}{K} = -\tan^{-1}\frac{X}{R}. \quad (302)$

[1] Bibliography, No. 5.

The general relationship between **I** and **E** in an alternating-current network may, therefore, be expressed in any of the forms:

$$\mathbf{I} = \frac{\mathbf{E}}{z} = \frac{\mathbf{E}}{\mathrm{R} + \mathrm{j}\mathrm{X}} = \frac{1}{z}\epsilon^{-\mathrm{j}\phi}\mathbf{E} \tag{303}$$

$$\mathbf{I} = y\mathbf{E} = (\mathrm{K} - \mathrm{jS})\mathbf{E} = y\epsilon^{-\mathrm{j}\phi}\mathbf{E}. \tag{304}$$

**39. Current Loci with Constant E.M.F. and Variable Impedance.**—Putting

$$\mathbf{E} = (\mathrm{R} + \mathrm{j}\mathrm{X})\mathbf{I}, \tag{305}$$

consider the result of keeping **E** and R constant, and varying X.

To eliminate the variable X from the equation, use will be made of the fact that

$$\mathrm{j}\mathbf{I} \cdot \mathbf{I} = 0. \tag{306}$$

Thus, taking the scalar product of the equation with **I**,

$$\mathbf{E} \cdot \mathbf{I} = \mathrm{R}\mathbf{I} \cdot \mathbf{I} + \mathrm{X}\mathrm{j}\mathbf{I} \cdot \mathbf{I}, \tag{307}$$

*i.e.*,

$$\mathbf{I} \cdot \mathbf{I} - \frac{\mathbf{E}}{\mathrm{R}} \cdot \mathbf{I} = 0. \tag{308}$$

Adding to each side

$$\frac{\mathbf{E} \cdot \mathbf{E}}{4\mathrm{R}^2} = \frac{\hat{\mathrm{E}}^2}{4\mathrm{R}^2}, \tag{309}$$

the result is

$$\mathbf{I} \cdot \mathbf{I} - 2\frac{\mathbf{I} \cdot \mathbf{E}}{2\mathrm{R}} + \frac{\mathbf{E} \cdot \mathbf{E}}{(2\mathrm{R})^2} = \frac{\hat{\mathrm{E}}^2}{4\mathrm{R}^2}, \tag{310}$$

*i.e.*,

$$\left(\mathbf{I} - \frac{\mathbf{E}}{2\mathrm{R}}\right)^2 = \left(\frac{\hat{\mathrm{E}}}{2\mathrm{R}}\right)^2 \tag{311}$$

$$= \text{const.} \tag{312}$$

The above equation means that the magnitude of the vector $\left(\mathbf{I} - \frac{\mathbf{E}}{2\mathrm{R}}\right)$ is constant, and is equal to $\frac{\hat{\mathrm{E}}}{2\mathrm{R}}$. Thus, as the impedance is varied by varying X and keeping R constant, the vector $\left(\mathbf{I} - \frac{\mathbf{E}}{2\mathrm{R}}\right)$ is always the radius of a constant circle. It will be seen by reference to Fig. 37 that the locus of the end of the **I** vector is this same circle of radius $\frac{\hat{\mathrm{E}}}{2\mathrm{R}}$.

In a precisely similar manner, by taking the scalar product of the original equation with j**I** instead of **I**, and thus eliminating R instead of X, it can be shown that, if the reactance is kept constant while the resistance is varied, the locus of the end of the current vector is a circle or radius $\frac{\hat{\mathrm{E}}}{2\mathrm{X}}$, the corresponding equation being

$$\left(\mathbf{I} - \frac{\mathrm{j}\mathbf{E}}{2\mathrm{X}}\right)^2 = \frac{\hat{\mathrm{E}}^2}{4\mathrm{X}^2} \tag{313}$$

$$= \text{const.} \tag{314}$$

Both of these loci are illustrated in Fig. 37. Students of electrical engineering will recognize in the above the basis of the circle diagrams of the induction motor.

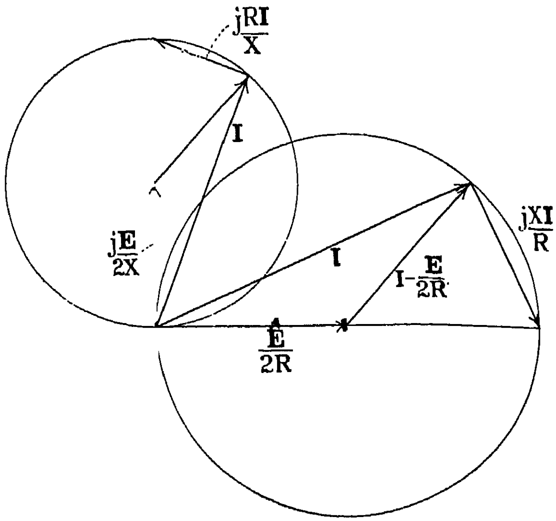


FIG. 37.

**40. Resonance.**—Considering the simple circuit of Par. 37, the reactance term X is given by

$$X = \omega L - \frac{1}{\omega C}. \tag{315}$$

If the values of L and C are such that

$$\omega L = \frac{1}{\omega C}, \tag{316}$$

*i.e.*,

$$\omega^2 = \frac{1}{LC}, \tag{317}$$

then

$$X = 0 \tag{318}$$

and

$$I = \frac{E}{R}. \tag{319}$$

The circuit is then said to be in a condition of resonance with the frequency of the supply e.m.f. Under these conditions the current reaches a maximum value

$$i = \frac{\hat{E}}{R} \cos wt \tag{320}$$

and is in phase with the e.m.f. In general, the vanishing of the X term will be taken as the definition of the condition of resonance,

and in all such cases the circuit will behave like a pure resistance. This does not necessarily mean that at resonance the current in the circuit will be the e.m.f. divided by the resistance of the circuit, since the resistance component may contain inductance and capacity terms and may, therefore, depend on the frequency. It does mean, however, that the current will be the e.m.f. divided by the effective resistance and will be in phase with the e.m.f.

It will be seen later that the condition

$$X = 0 \tag{321}$$

may be satisfied by more than one value of the frequency. The corresponding current values in such a case will be critical or turning values, *i.e.*, either maxima or minima.

**41. Energy Conditions in Alternating-current Networks.**[1]— A current of instantaneous magnitude i (amperes) in falling through a potential e (volts) gives up ie joules of energy per second, *i.e.*, the instantaneous energy rate is p = ie watts.

If
$$e = \mathbf{E} \cdot \boldsymbol{\nu} = \hat{E} \cos \omega t \tag{322}$$
and
$$i = \mathbf{I} \cdot \boldsymbol{\nu} = \hat{I} \cos (\omega t - \phi), \tag{323}$$
the instantaneous power is
$$p = (\mathbf{I} \cdot \boldsymbol{\nu})(\mathbf{E} \cdot \boldsymbol{\nu}) \tag{324}$$
and, as shown in Par. 32, this may be expressed in the form
$$p = \frac{\mathbf{I} \cdot \mathbf{E}}{2} + \mathbf{W} \cdot \boldsymbol{\nu}, \tag{325}$$
where **W** is a vector of magnitude $\frac{\hat{I}\hat{E}}{2}$ making with $\boldsymbol{\nu}$ an angle equal to the sum of the angles made by **I** and **E** with $\nu$, *i.e.*,
$$\mathbf{W} \cdot \boldsymbol{\nu} = \frac{\hat{I} \cdot \hat{E}}{2} \cos (2\omega t - \phi). \tag{326}$$

Thus, the instantaneous rate at which energy is being given up by the current **I** in falling through the potential **E** consists of two parts, one being a constant term $\frac{\mathbf{I} \cdot \mathbf{E}}{2}$, the constant value of which is $\left(\frac{\hat{I}\hat{E}}{2}\right) \cos \phi$, and the other a periodic double-frequency term $\mathbf{W} \cdot \boldsymbol{\nu}$. Since the mean value of the latter over a period is zero, then for P, the mean rate at which energy is being given up by the current (*i.e.*, the mean power), the value is

[1] Bibliography, No. 4.

$$P = \frac{1}{T}\int_0^T p\,dt = \frac{\mathbf{I}\cdot\mathbf{E}}{2}. \tag{327}$$

For the simple circuit considered in Par. 37 it is easy to show that the whole of the power being consumed in the circuit is associated with the resistance.

Since $$\mathbf{E} = (R + jX)\mathbf{I} \tag{328}$$

therefore $$P = \frac{\{(R + jX)\mathbf{I}\cdot\mathbf{I}\}}{2} \tag{329}$$

$$= R\frac{\mathbf{I}\cdot\mathbf{I}}{2} \tag{330}$$

since $$jX\mathbf{I}\cdot\mathbf{I} = 0. \tag{331}$$

It was shown in Par. 21 that

$$\mathbf{I}\cdot\mathbf{I} = \hat{I}^2, \tag{332}$$

so that $$P = \frac{R\hat{I}^2}{2}. \tag{333}$$

Putting I for the root-mean-square value of the alternating current, *i.e.*,

$$I^2 = \frac{\hat{I}^2}{2} \tag{334}$$

$$P = RI^2. \tag{335}$$

Thus, the resistance is the only part of the circuit in which energy is actually being consumed. In the other parts of the circuit, energy is periodically being absorbed and given out.

The process can be followed in detail by applying the energy equation to the three elements of the circuit separately. Thus, for $p_L$, the instantaneous rate of change of the energy associated with the inductance L,

$$p_L = \frac{\mathbf{I}\cdot\mathbf{E}_L}{2} + \mathbf{W}_L\cdot\mathbf{v} \tag{336}$$

$$= \frac{\omega L\hat{I}^2}{2}\cos\left(2\omega t - \frac{\pi}{2}\right) = \frac{-\omega L\hat{I}^2}{2}\sin 2\omega t, \tag{337}$$

since $$\mathbf{I}\cdot\mathbf{E}_L = 0. \tag{338}$$

Similarly, for $p_C$, the instantaneous rate of change of the energy associated with the capacity,

$$p_C = \frac{\mathbf{I}\cdot\mathbf{E}_C}{2} + \mathbf{W}_C\cdot\mathbf{v} \tag{339}$$

$$= \frac{\hat{I}^2}{2\omega C}\sin(2\omega t). \tag{340}$$

Further, $p_R$ is given by $$p_R = \frac{\mathbf{I}\cdot\mathbf{E}_R}{2} + \mathbf{W}_R\cdot\mathbf{v}, \tag{341}$$

*i.e.*, $$p_R = \frac{\hat{I}^2R}{2} + \frac{\hat{I}^2R}{2}\cos 2\omega t. \tag{342}$$

Therefore, the total instantaneous rate of energy supply to the circuit is

$$p = \frac{\hat{I}^2}{2}\{R + R\cos 2\omega t + \left(\frac{1}{\omega C} - \omega L\right)\sin 2\omega t\}\cdot \tag{343}$$

Thus, it appears that the energy change associated with the capacity is opposite in phase to that associated with the inductance. While the one is absorbing energy, the other is giving out energy, and *vice versa.* At resonance $\omega L = \frac{1}{\omega C}$, so that the capacity and the inductance form, as it were, a self-supporting system, the one supplying energy at the same rate as the other absorbs it.

In general, for a current I and a potential E such that

$$I = \frac{E}{(R + jX)} = \frac{1}{z}\epsilon^{-j\phi}E \tag{344}$$

or

$$I = (K - jS)E = y\epsilon^{-j\phi}E, \tag{345}$$

the mean power is

$$P = \frac{I \cdot E}{2} = \left(\frac{\hat{I}\hat{E}}{2}\right)\cos\phi \tag{346}$$

$$= \frac{R\hat{I}^2}{2} \tag{347}$$

$$= RI^2 \tag{348}$$

$$= \frac{K\hat{E}^2}{2} \tag{349}$$

$$= KE^2 \tag{350}$$

in which R and K are, respectively, the resistance component of the general impedance operator and the conductance component of the general admittance operator.

**42. The General Case of Impedances in Parallel.**—Let

$$z_1 = R_1 + jX_1 \tag{351}$$

$$z_2 = R_2 + jX_2 \tag{352}$$

$$z_3 = R_3 + jX_3 \tag{353}$$

etc., etc.

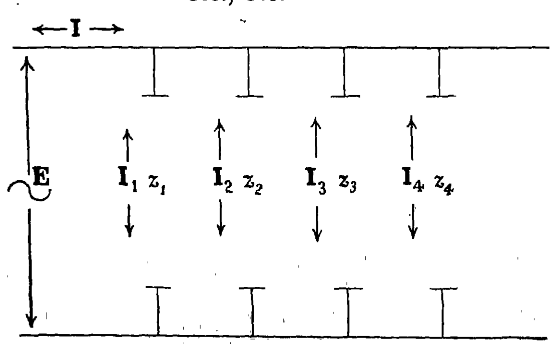


FIG. 38.

be a number of impedances connected in parallel, as shown in Fig. 38. If $i_1$, $i_2$, $i_3$, etc., represented by the vectors $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$, etc., be the currents flowing in the various circuits, then

$$\mathbf{E} = z_1\mathbf{I}_1 = z_2\mathbf{I}_2 = z_3\mathbf{I}_3 = \text{etc., etc.} \tag{354}$$

*i.e.*,

$$\mathbf{I}_1 = \frac{\mathbf{E}}{z_1} \tag{355}$$

$$\mathbf{I}_2 = \frac{\mathbf{E}}{z_2} \tag{356}$$

$$\mathbf{I}_3 = \frac{\mathbf{E}}{z_3} \tag{357}$$

etc., etc.

Further, if the total current be represented by $\mathbf{I}$, and the total impedance by $z$,

$$\mathbf{I} = \frac{\mathbf{E}}{z}. \tag{358}$$

By Kirchhoff's first law,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \text{etc., etc.} \tag{359}$$

Therefore, from Eqs. (355) to (359) inclusive

$$\frac{\mathbf{E}}{z} = \frac{\mathbf{E}}{z_1} + \frac{\mathbf{E}}{z_2} + \frac{\mathbf{E}}{z_3} + \frac{\mathbf{E}}{z_4} + \text{etc., etc.} \tag{360}$$

so that

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} + \text{etc., etc.} \tag{361}$$

*i.e.*,

$$y = y_1 + y_2 + y_3 + y_4 + \text{etc., etc.} \tag{362}$$

Thus, impedances in parallel combine in a similar manner to resistances in parallel.

**43. Capacity and Resistance in Parallel.**—As an example of the above, consider the case illustrated in Fig. 39, *i.e.*, a condenser of capacity C shunted by a resistance R. For the combined impedance

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}, \tag{363}$$

*i.e.*,

$$z = \frac{z_1 z_2}{(z_1 + z_2)}, \tag{364}$$

where

$$z_1 = \mathrm{R} \tag{365}$$

and

$$z_2 = \frac{1}{j\omega \mathrm{C}} = \frac{-j}{\omega \mathrm{C}}. \tag{366}$$

Therefore,
$$z = \frac{\dfrac{-Rj}{\omega C}}{R - \dfrac{j}{\omega C}} \tag{367}$$
$$= \frac{-jR}{(\omega CR - j)} \tag{368}$$
$$= \frac{R}{1 + \omega^2C^2R^2} + \frac{1}{j\omega C\left(1 + \dfrac{1}{\omega^2C^2R^2}\right)} \tag{369}$$

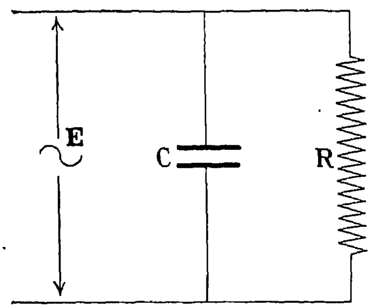


FIG. 39.

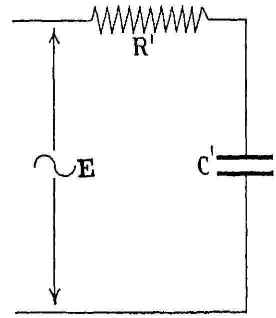


FIG. 40.

The circuit of Fig. 39 can, therefore, be replaced by the equivalent circuit shown in Fig. 40, in which
$$R' = \frac{R}{1 + \omega^2C^2R^2} \tag{370}$$
and
$$C' = C\left(1 + \frac{1}{\omega^2C^2R^2}\right). \tag{371}$$

The above equations are of considerable practical significance For instance, suppose R to be the insulation resistance of the condenser or the shunt resistance equivalent to its dielectric losses. Then from the above equations it is seen that the effect of imperfect insulation or dielectric losses in the condenser is virtually to introduce into the circuit a small series resistance In low or audible frequencies the effect may be quite inappreciable, but at radio frequencies it may be considerable. For instance, let
$$\omega = 10^6,$$
*i.e.*, the frequency is $\frac{10^6}{2\pi}$, and let $R = 5 \times 10^6$ ohms. Then, taking C as 100 micro-microfarads, the effective resistance introduced into the circuit is

$$R' = \frac{5 \times 10^6}{1 + 10^{12} \cdot 10^{-20} \cdot 10^{12} \cdot 25} \omega$$
$$= 20 \text{ ohms.}$$

Under these conditions, therefore, an insulation resistance of 5 megohms may materially increase the effective resistance of the circuit. This emphasizes the importance of good insulation at radio frequencies.

Again, suppose the circuit shown in Fig. 41 to represent a wavemeter, the resonance of which is indicated by a detector of some kind connected across the condenser, and suppose the resistance of the detector to be of the order of 25,000 ohms. Then, the other quantities remaining as before, for $100 \frac{(C' - C)}{C}$, the percentage change of capacity attributable

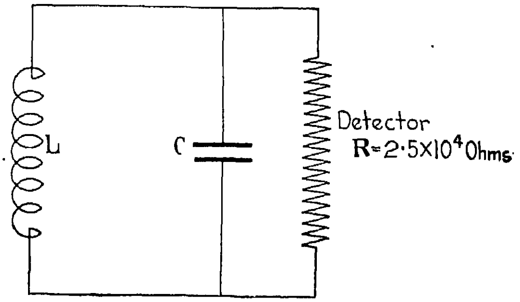


FIG. 41.

to the shunt resistance,

$$\frac{100}{\omega^2 C^2 R^2} = \frac{10^2}{10^{12} 10^{-10} 10^8 \times 6 \cdot 25}$$
$$= 16.$$

Thus there is a 16 per cent change in the condenser reading and an error of 8 per cent in the calculated value of $\lambda$, the wave length (since $\lambda \propto \sqrt{C}$).

**44. Inductance and Resistance in Parallel with a Capacity and a Resistance.**—This case, illustrated in Fig. 42, is of considerable theoretical interest.

Putting $z_1$ for the impedance of the inductive branch,

$$z_1 = (R_1 + \omega jL) \tag{372}$$

and for the corresponding admittance $y_1$,

$$y_1 = \frac{R_1}{z_1^2} - \frac{j\omega L}{z_1^2}. \tag{373}$$

Similarly, putting $z_2$ for the impedance of the condenser branch

$$z_2 = R_2 - \frac{j}{\omega C}, \qquad (3$$

so that
$$z_2^2 = R_2^2 + \frac{1}{\omega^2 C^2} \qquad (3$$

and
$$y_2 = \frac{R_2}{z_2^2} + \frac{\frac{j}{\omega C}}{z_2^2}. \qquad (3$$

For $y$, the admittance of the two circuits in parallel,

$$y = y_1 + y_2, \qquad (3$$

*i.e.*,
$$y = \frac{R_1}{z_1} + \frac{R_2}{z_2} + j\left\{\frac{\frac{1}{\omega C}}{z_2} - \frac{\omega L}{z_1}\right\} \qquad (3$$

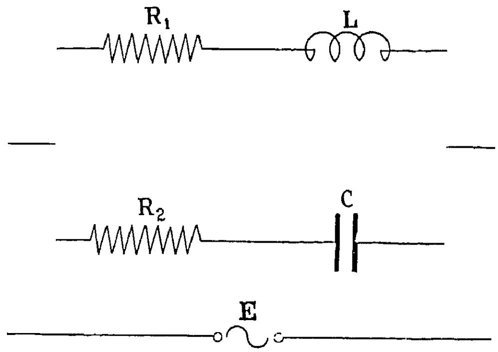


FIG. 42.

Consider now the condition that the joint admittance (and, therefore, t joint impedance) shall contain no $j$ term,

*i.e.*,
$$\frac{\frac{1}{\omega C}}{z_2} = \frac{\omega L}{z_1}, \qquad (37$$

or
$$\omega^2 LC\left(R_2^2 + \frac{1}{\omega^2 C^2}\right) = R_1^2 + \omega^2 L^2. \qquad (38$$

The resonant frequency in the general case is, therefore, given by

$$\omega^2 = \frac{1}{LC}\,\frac{R_1^2 - \frac{L}{C}}{R_2^2 - \frac{L}{C}}. \qquad (38$$

If
$$R_1 = R_2 = R, \qquad (38$$

this becomes
$$\omega^2 = \frac{1}{LC}. \qquad (38$$

If, however, in addition to $R_1 = R_2 = R$, (38

$$L = CR^2, \qquad (38$$

then Eq. (380) is satisfied for all values of $\omega$, *i.e.*, for all values of the frequency. This circuit, therefore, under these conditions, possesses the remarkable property of behaving like a non-inductive resistance at all frequencies, the value of this resistance being R, as can be seen by considering the case when $\omega = 0$.

**45. Inductively Coupled Circuits.**—Consideration will now be given to the general case illustrated in Fig. 43, which represents two circuits of the type considered in Par. 37, the inductances of which have a mutual inductance M. In series with the first circuit is a source of alternating e.m.f.,

$$e = \mathbf{E} \cdot \boldsymbol{\nu} = \hat{E} \cos \omega t. \tag{386}$$

As will be shown in a later section, both of these circuits will give two free oscillations at the moment the source of e.m.f. is intro-

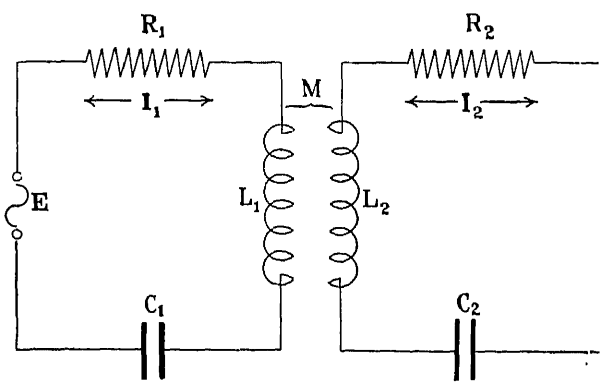


FIG. 43.

duced into the first circuit. For the present, however, only the quasi-stationary condition which persists when the amplitudes of the free oscillations have become inappreciable will be considered.

First, however, it will be necessary to consider the vectorial form of the e.m.f. due to mutual induction. This can be derived directly from the definition of mutual induction. The e.m.f. induced in the second circuit by the current $i_1$ in the first circuit is given by

$$e_{m2} = -M\frac{di_1}{dt}. \tag{387}$$

In vector form this becomes

$$\mathbf{E}_{m2} = -M\omega j\mathbf{I}_1. \tag{388}$$

Similarly, the e.m.f. induced in the first circuit by the current $i_2$ flowing in second circuit is given by

$$E_{m1} = -M\omega jI_2. \tag{389}$$

Applying Kirchhoff's second law to the first circuit the result is, therefore,

$$E + E_{R1} + E_{L1} + E_{C1} + E_{M1} = 0, \tag{390}$$

and for the second circuit

$$E_{R2} + E_{L2} + E_{C2} + E_{M2} = 0. \tag{391}$$

Expressing these potential differences in terms of the current vectors the result is

$$\left(R_1 + \omega jL_1 + \frac{1}{\omega jC_1}\right)I_1 + M\omega jI_2 = E \tag{392}$$

for the first circuit, and

$$\left(R_2 + \omega jL_2 + \frac{1}{\omega jC_2}\right)I_2 + M\omega jI_1 = 0 \tag{393}$$

for the second.

Putting

$$z_1 = R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right) = R_1 + jX_1 \tag{394}$$

and $$z_2 = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right) = R_2 + jX_2, \tag{395}$$

the equations can be written

$$(R_1 + jX_1)I_1 + M\omega jI_2 = E \tag{396}$$
$$(R_2 + jX_2)I_2 + M\omega jI_1 = 0. \tag{397}$$

From the second of these,

$$I_2 = -\frac{M\omega j}{R_2 + jX_2} I_1 \tag{398}$$
$$= -M\omega j\frac{(R_2 - jX_2)}{z_2^2} I_1. \tag{399}$$

Substituting in the first equation this value of $I_2$ in terms of $I_1$

$$\left\{(R_1 + jX_1) + M^2\omega^2\frac{(R_2 - jX_2)}{z_2^2}\right\}I_1 = E, \tag{400}$$

$$\left\{\left(R_1 + \frac{M^2\omega^2R_2}{z_2^2}\right) + j\left(X_1 - \frac{M^2\omega^2X_2}{z_2^2}\right)\right\}I_1 = E. \tag{401}$$

This gives the primary current in terms of the primary e.m.f and an effective impedance. The form of this effective impedance makes clear the effect of the secondary circuit on the primary circuit. It is seen, in fact, that the resistance of the primary circuit is increased by an amount of $M^2\omega^2$ times the conductance

of the secondary circuit, and that the reactance of the primary circuit is decreased by an amount of $M^2\omega^2$ times the susceptance of the secondary circuit.

Putting the effective primary impedance in the form

$$R_e + jX_e = \left(R_1 + \frac{M^2\omega^2 R_2}{R^2 + X^2}\right) + j\left(X_1 - \frac{M^2\omega^2 X_2}{R_2^2 + X_2^2}\right), \quad (402)$$

the effect of various special conditions can now be considered.

1. *Secondary Resistance Negligible.*—Putting $R_2 = 0$, then

$$R_e = R_1 \quad (403)$$

and

$$X_e = X_1 - \frac{M^2\omega^2}{X_2}. \quad (404)$$

If, now, $X_2 = 0$, *i.e.*, if the secondary circuit is tuned to the frequency of the primary e.m.f., then $X_e$ becomes infinite.

2. *Resonance Condition.*—The condition for resonance in the primary circuit is

$$X_e = 0, \quad (405)$$

which gives

$$X_1(R_2^2 + X_2^2) = M^2\omega^2 X_2, \quad (406)$$

*i.e.*,

$$\left(\omega L_1 - \frac{1}{\omega C_1}\right)\left\{R_2^2 + \left(\omega L_2 - \frac{1}{\omega C_2}\right)^2\right\} = M^2\omega^2\left(\omega L_2 - \frac{1}{\omega C_2}\right). \quad (407)$$

This is in the form of a cubic equation in $\omega$, which indicates that there are, in general, three frequencies at which the primary circuit will behave as a non-inductive resistance. The solution of the cubic equation in the general case can be found by the usual methods.

If the resistance of the secondary circuit is negligibly small, the equation becomes a quadratic, yielding a simple solution. Putting $R_2 = 0$ in Eq. (406), the result is

$$X_1 X_2 = M^2\omega^2, \quad (408)$$

*i.e.*,

$$\left(\omega L_1 - \frac{1}{\omega C_1}\right)\left(\omega L_2 - \frac{1}{\omega C_2}\right) = M^2\omega^2. \quad (409)$$

Dividing through by $\frac{1}{L_1 L_2}$ and putting

$$\frac{1}{L_1 C_1} = \omega_1^2 \quad (410)$$

$$\frac{1}{L_2 C_2} = \omega_2^2 \quad (411)$$

$$\frac{M^2}{L_1 L_2} = k^2, \quad (412)$$

the equation becomes

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = k^2\omega^4, \tag{413}$$

of which the solution by the usual methods will be found to be

$$\omega^2 = \frac{(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2) - 4\omega_1^2\omega_2^2(1 - k^2)}}{2(1 - k^2)}. \tag{414}$$

If the circuits are isochronous, *i.e.*, if

$$\omega_1 = \omega_2 = \omega_0, \tag{415}$$

the above expression reduces to

$$\omega^2 = \frac{1 \pm k}{1 - k^2}\omega_0^2 \tag{416}$$

and for the two resonant frequencies the result is

$$\omega' = \frac{1}{\sqrt{1 - k}}\omega_0 \text{ and } \omega'' = \frac{1}{\sqrt{1 + k}}\omega_0. \tag{417}$$

This result can be compared with that for the corresponding free periods of two coupled circuits given in Par. 63.

3. *The Alternating-current Transformer*.—The elementary theory of the alternating-current transformer is the special case of the foregoing, illustrated in Fig. 44, when the circuits do not

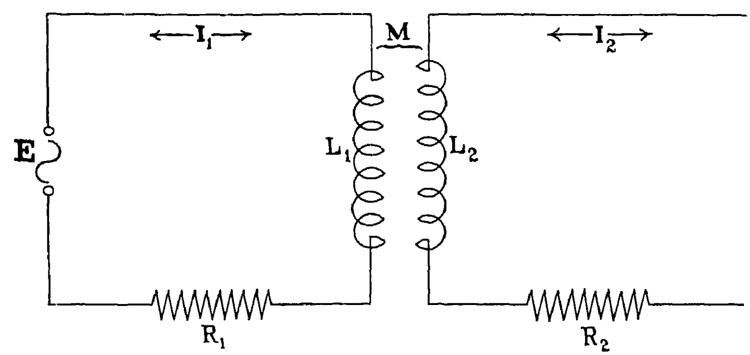


FIG. 44.

include the condensers. Putting $C_1 = C_2 = \infty$ in the equations of the general case, then

$$X_1 = \omega L_1 \tag{418}$$

$$X_2 = \omega L_2, \tag{419}$$

so that

$$I_2 = -\frac{M\omega j}{R_2 + j\omega L_2} I_1 \tag{420}$$

$$= -\frac{M\omega j}{z_2} I_1 \tag{421}$$

and $$\mathbf{I}_1 = \frac{\mathbf{E}}{(R_e + j\omega L_e)}, \tag{422}$$

where $$R_e = R_1 + \frac{M^2\omega^2 R_2}{z_2^2} \tag{423}$$

and $$L_e = L_1 - \frac{M^2\omega^2 L_2}{z_2^2}. \tag{424}$$

Thus, the resistance of the primary circuit is apparently increased and its inductance apparently decreased by $\frac{M^2\omega^2}{z_2^2}$ times the corresponding quantities of the secondary circuit. The primary circuit can, in fact, be represented by the circuit shown in Fig. 45, the negative inductance being represented by an equivalent capacity.

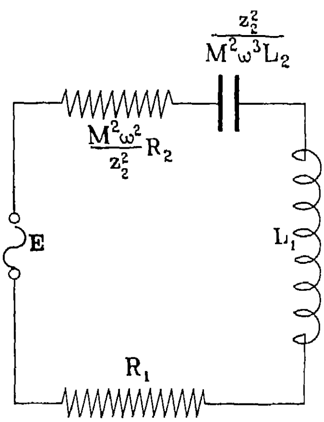


FIG. 45.

From the above vector equations all the chief features of the elementary theory of the transformer can be readily deduced. As examples, the effect of a low-resistance secondary circuit, and the principle of the constant-ratio current transformer will be considered.

(*a*) The Effect of a Low-resistance Secondary Circuit.—If $R_2$ is very small compared with $L_2$, then

$$L_e = L_1 - \frac{M^2}{L_2} \tag{425}$$

$$= L_1\left(1 - \frac{M^2}{L_1 L_2}\right). \tag{426}$$

The quantity $\frac{M}{\sqrt{L_1 L_2}}$ is known as the "coefficient of coupling." Calling this k, then

$$L_e = L_1(1 - k^2). \tag{427}$$

The theoretical maximum value of k is unity. For this value

$$L_e = 0. \tag{428}$$

This condition cannot actually be realized in practice, but it is obvious that, in general, the effect of a closely coupled secondary circuit of low resistance will be to reduce considerably the effective inductance of the primary circuit.

(*b*) The Constant-ratio Current Transformer.—Referring to Eq. (399), if $R_2$ is negligible compared with $L_2$, then

$$I_2 = -\frac{M}{L_2}I_1, \tag{429}$$

*i.e.*, the currents in the secondary and in the primary circuits are related by a constant multiple which is independent of frequency. This fact is utilized in the construction of current transformers for measurement purposes. The arrangement is particularly suitable for use at radio frequencies, when $R_2$ is easily made quite negligible compared with $L_2$.[1]

**46. The Application of Vector Analysis to Alternating-current Bridge Circuits.**—The circuit shown in Fig. 46 may be taken as

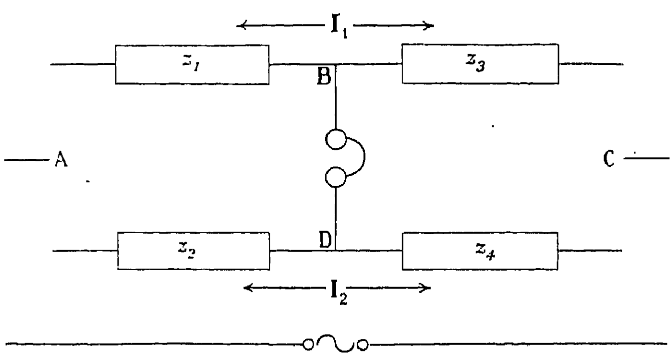


Fig. 46.

typical of a large number of alternating-current bridge circuits $z_1$, $z_2$, $z_3$, and $z_4$ being impedances of the general type

$$z_1 = R_1 + jX_1 \tag{430}$$
$$z_2 = R_2 + jX_2 \tag{431}$$
$$z_3 = R_3 + jX_3 \tag{432}$$
$$z_4 = R_4 + jX_4. \tag{433}$$

In most practical cases, two of these, say, $z_1$ and $z_2$, are fixed in value and constitute what is known as the "ratio arms" of the bridge. Here $z_3$ will be considered to be the unknown impedance, the components of which are to be determined, and $z_4$ a variable impedance, the resistance and the reactance components of which can be varied until balance is obtained, this condition being indicated by silence in the telephones or by any other suitable means. In this balanced condition let $I_1$ and $I_2$

[1] See paper by Campbell and Dye, *Proc. Roy. Soc.*, on this subject.

be vectors representing the currents flowing in the upper and the lower arms of the bridge respectively. Then

$$\mathbf{I}_1(z_1 + z_3) = \mathbf{I}_2(z_2 + z_4). \tag{434}$$

Also, since there is no potential difference between the points B and D (this being the definition of the balance condition),

$$\mathbf{I}_1 z_1 = \mathbf{I}_2 z_2. \tag{435}$$

Therefore,

$$\frac{z_1}{(z_1 + z_3)} = \frac{z_2}{(z_2 + z_4)} \tag{436}$$

or

$$\frac{z_1}{z_3} = \frac{z_2}{z_4}, \tag{437}$$

*i.e.*,

$$\frac{R_1 + jX_1}{R_3 + jX_3} = \frac{R_2 + jX_2}{R_4 + jX_4}, \tag{438}$$

so that $(R_1R_4 - X_1X_4) + j(R_1X_4 + X_1R_4)$

$$= (R_2R_3 - X_2X_3) + j(R_2X_3 + X_2R_3). \tag{439}$$

Therefore, by Par. 15

$$R_1R_4 - X_1X_4 = R_2R_3 - X_2X_3 \tag{440}$$

$$R_1X_4 + X_1R_4 = R_2X_3 + X_2R_3. \tag{441}$$

Thus there are in general, two conditions to be fulfilled simultaneously. It is for this reason that nearly all forms of alternating-current bridges involve a double balance, $R_4$ and $X_4$ being varied successively until complete balance is obtained.

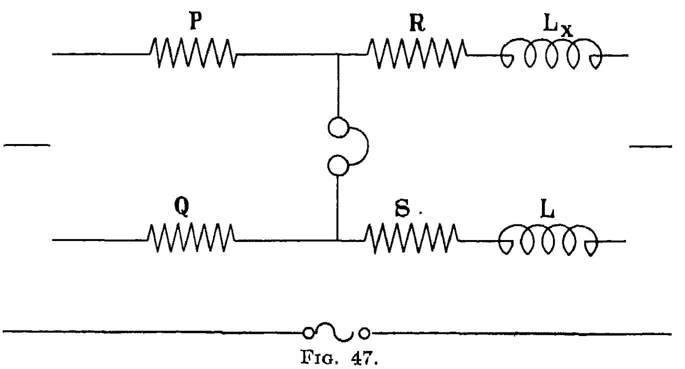


FIG. 47.

As examples of the application of the general balance condition, two typical cases will be considered:

1. The simple inductance bridge (Fig. 47).

2. The Max Wien bridge for the measurement of equivalent shunt resistance of the losses in a condenser (Fig. 48).

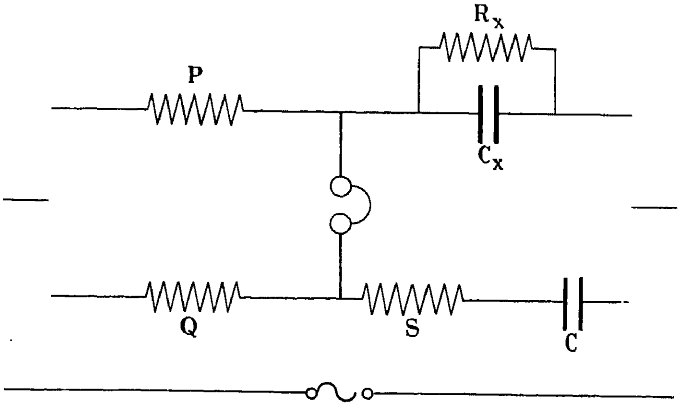


FIG. 48.

1. In this case

$$R_1 = P \text{ and } X_1 = 0 \tag{442}$$
$$R_2 = Q \qquad X_2 = \omega L_x \tag{443}$$
$$R_3 = R \qquad X_3 = 0 \tag{444}$$
$$R_4 = S \qquad X_4 = \omega L. \tag{445}$$

The conditions for balance are, therefore,

$$PS = QR \tag{446}$$

and

$$\omega LP = \omega L_x R, \tag{447}$$

*i.e.*,

$$\frac{P}{R} = \frac{Q}{S} = \frac{L_x}{L}. \tag{448}$$

2. In the Max Wien bridge,

$$R_1 = P \text{ and } X_1 = 0 \tag{449}$$
$$R_3 = Q \qquad X_3 = 0 \tag{450}$$
$$R_4 = S \qquad X_4 = \frac{1}{\omega C} \tag{451}$$

and, from Eqs. (370) and (371),

$$R_2 = \frac{R_x}{(1 + \omega^2 C_x^2 R_x^2)} \text{ and } X_2 = -\frac{1}{\omega C_x\left(1 + \dfrac{1}{\omega^2 C_x^2 R_x^2}\right)}. \tag{452}$$

The conditions for balance are, therefore,

$$PS = \frac{R_x Q}{1 + \omega^2 C_x^2 R_x^2}, \tag{453}$$

$$-\frac{P}{\omega C} = \frac{Q}{\omega C_x\left(1 + \dfrac{1}{\omega^2 C_x^2 R_x^2}\right)}, \tag{454}$$

*i.e.*

$$PS = \frac{QR_x}{1 + \omega^2 C_x^2 R_x^2} \tag{455}$$

$$\frac{P}{\omega C} = \frac{\omega C_x Q R_x^2}{1 + \omega^2 C_x^2 R_x^2}. \tag{456}$$

For the determination of $R_x$ it is unnecessary to know the value of $C_x$, since the latter can be eliminated from the above two equations. Dividing the first by the second

$$S\omega C = \frac{1}{\omega C_x R_x}. \tag{457}$$

Therefore,

$$\omega_x{}^2 C_x{}^2 R_x{}^2 = \frac{1}{S^2\omega^2 C^2}. \tag{458}$$

Substituting this in the first equation

$$PS = \frac{QR_x}{1 + \dfrac{1}{S^2\omega^2C^2}}. \tag{459}$$

Therefore,

$$R_x = \frac{P(1 + \omega^2C^2S^2)}{Q\omega^2C^2S}. \tag{460}$$

Examples of bridge circuits such as the above could, of course, be multiplied almost indefinitely. The two given are, however, sufficient for the present purpose, which is to illustrate the application of vectorial methods to their analysis.[1]

## EXAMPLES

**1.** Given the following conductors:

(*A*) (1) A pure resistance of 500 ohms.

(2) A pure capacity of 6.366 microfarads.

(3) A pure inductance of 3.183 henries.

Express in the forms $r\epsilon^{j\theta}$ and $(a + jb)$ the impedances and the admittances of the following combinations of the conductors at a frequency of 50 p.p.s.:

(*a*) (1).
(*b*) (2).
(*c*) (3).
(*d*) (1) and (2) in series.
(*e*) (2) and (3) in series.
(*f*) (3) and (1) in series.
(*g*) (1), (2), and (3) in series.
(*h*) (1) and (2) in parallel.
(*k*) (2) and (3) in parallel.
(*l*) (3) and (1) in parallel.
(*m*) (1), (2), and (3) in parallel.
(*n*) (2), in parallel with (1) and (3) in series.

(*B*) What happens in cases e, g, k, m, and n if the capacity is reduced to 3.183 microfarads?

**2.** Draw the complete vector diagram for a circuit consisting of:

(*a*) A pure resistance of 5 ohms in series with a pure capacity of 318.3 microfarads in series with a pure inductance of 63.66 millihenries in series with a 50-cycle e.m.f. of 100 volts.

(*b*) The same conductors in parallel, in series with the same e.m.f.

[1] A complete account of alternating-current bridge measuremer*~ analyzed or vectorial lines, will be found in ref. No. 19, Bibliography.

**3.** A pure resistance R is in parallel with a pure reactance X.

(*a*) If R is kept constant while X is varied from $-\infty$ to $+\infty$, prove that the impedance of the two in parallel can be represented by a line such that, one end being fixed in position, the other end moves around a circle of diameter R.

(*b*) If X is kept constant while R is varied from 0 to $\infty$, prove that the impedance of the two in parallel can be represented by a line such that, one end being fixed in position, the other end moves around a semicircle of diameter X.

**4.** An e.m.f. of root-mean-square value 70.7 volts supplies current of 50-cycle frequency to a circuit consisting of a pure resistance of 10 ohms in series with a pure capacity of 31.83 microfarads in series with a pure inductance of 159.15 millihenries.

(*a*) Give, in scalar form, the expressions for:

(1) The instantaneous rate at which electrical energy is being absorbed by or given out by each element of the circuit separately.

(2) The instantaneous rate at which energy is being supplied to the whole circuit.

(3) The mean value over a period of the rate at which energy is being absorbed by or given out by each element of the circuit separately.

(4) The mean value over a period of the rate at which energy is being supplied to the circuit as a whole.

(*b*) Give the same expressions for the case in which the capacity is reduced to 15.915 microfarads.

**5.** Two air-cored coils have inductances of 5 and 50 henries respectively, with resistances of 10 and 10,000 ohms respectively. The mutual inductance between the coils is 11.18 henries.

An e.m.f. of 100 volts is in series with the first of these coils. The frequency of the e.m.f. is 79.6 p.p.s. Give, in scalar form, the expressions for the instantaneous values of the currents flowing in the two circuits, taking the origin of time as the instant when the e.m.f. is at its maximum value.

Give also, for comparison, the expression for the instantaneous value of the current in the first circuit when the resistance of the second circuit is made infinite.

## ANSWERS TO EXAMPLES

**1.** (*A*)

| | *Impedances* | *Admittances* |
|---|---|---|
| (*a*) | $500 + 0j$; $500\epsilon^{j0}$. | $.002 + 0j$; $.002\epsilon^{j0}$. |
| (*b*) | $0 - 500j$; $500\epsilon^{-j90°}$. | $0 + .002j$; $.002\epsilon^{j90°}$. |
| (*c*) | $0 + 1{,}000j$; $1{,}000\epsilon^{j90°}$. | $0 - .001j$; $.001\epsilon^{-j90°}$. |
| (*d*) | $500 - 500j$; $707\epsilon^{-j45°}$. | $.001 + .001j$; $.00141\epsilon^{j45°}$. |
| (*e*) | $0 + 500j$; $500\epsilon^{j90°}$. | $0 - .002j$; $.002\epsilon^{-j90°}$. |
| (*f*) | $500 + 1{,}000j$; $1{,}118\epsilon^{j63°26'}$. | $.0004 - .0008j$; $.000894\epsilon^{-j63°26'}$. |
| (*g*) | $500 + 500j$; $707\epsilon^{j45°}$. | $.001 - .001j$; $.00141\epsilon^{-j45°}$. |
| (*h*) | $250 - 250j$; $354\epsilon^{-j45°}$. | $.002 + .002j$; $.00283\epsilon^{j45°}$. |

(*k*) $0 - 1{,}000j$; $1{,}000\epsilon^{-j90°}$. $0 + .001j$; $.001\epsilon^{j90°}$.
(*l*) $400 + 200j$; $447.2\epsilon^{j26°34'}$. $.002 - .001j$; $.002236\epsilon^{-j26°34'}$.
(*m*) $400 - 200j$; $447.2\epsilon^{-j26°34'}$. $.002 + .001j$; $.002236\epsilon^{j26°34'}$.
(*n*) $250 - 750j$; $790.5\epsilon^{-j71°33'}$. $.0004 + .0012j$; $.001265\epsilon^{j71°33'}$.
(*B*)
$0$; $0$. $\infty$; $\infty$.
$500 + 0j$; $500\epsilon^{j0}$. $.002 + 0j$; $.002\epsilon^{j0}$.
$\infty$; $\infty$. $0$; $0$.
$500 + 0j$; $500\epsilon^{j0}$. $.002 + 0j$; $.002\epsilon^{j0}$.
$2{,}000 - 1{,}000j$; $2{,}236\epsilon^{-j26°34'}$. $.0004 + .0002j$; $.000447\epsilon^{j26°34'}$.

**2.** Representing the e.m.f. by the vector $141{\cdot}4\mathbf{v}$, then the other vectors of the diagram will be:

(*a*)
$$E_R = 63.15\epsilon^{j116°34'}\ \mathbf{v}.$$
$$E_L = 253\epsilon^{j153°26'}\ \mathbf{v}.$$
$$E_C = 126.3\epsilon^{j26°34'}\ \mathbf{v}.$$
$$I = 12.63\epsilon^{-j63°26'}\ \mathbf{v}.$$

(*b*)
$$I = 29.2\epsilon^{j14°2'}\ \mathbf{v}.$$
$$I_R = 28.3\mathbf{v}.$$
$$I_L = -7.07j\mathbf{v}.$$
$$I_C = 14.14j\mathbf{v}.$$

**3.** Outline of proof for first part:

$$\frac{\mathbf{v}}{z} = \frac{\mathbf{v}}{R} - \frac{j\mathbf{v}}{X}$$
$$\mathbf{v} = \frac{1}{R}z\mathbf{v} - \frac{1}{X}jz\mathbf{v}.$$

Taking the scalar product with $z\mathbf{v}$,

$$\frac{z\mathbf{v}\cdot z\mathbf{v}}{R} = \mathbf{v}\cdot z\mathbf{v},$$

*i.e.*,
$$\left(z\mathbf{v} - \frac{R\mathbf{v}}{2}\right)^2 = \frac{R^2\mathbf{v}^2}{4} = \frac{R^2}{4}.$$

**4.** (*a*) (1) $p_L = 192.31 \cos (628t + 112° 36')$ watts
$p_C = -96.15 \cos (628t + 112° 36')$ watts
$p_R = 19.23 + 19.23 \cos (628t + 28° 36')$ watts.

(2) $p = 19.23 + 19.23 \cos (628t + 22° 36') + 96.15 \cos (628t + 112° 36')$ watts.

(3) $P_L = 0$
$P_C = 0$
$P_R = 19.305$ watts

(4) $P = 19.305$ watts.

(*b*) (1) $p_L = 5{,}000 \cos (628t - 90°)$ watts
$p_C = -5{,}000 \cos (628t - 90°)$ watts
$p_R = 500 + 500 \cos 628t$ watts.

(2) $p = 500 + 500 \cos 628t$ watts.

(3) $P_L = 0$
$P_C = 0.$
$P_R = 500$ watts

(4) $P = 500$ watts.

**5.** Current in first circuit:

$$.0951 \cos (500t - 72^\circ\ 45').$$

Current in the second circuit:

$$.01976 \cos (500t + 129^\circ\ 3').$$

If the second circuit is open, the current in the first circuit will be:

$$.0566 \cos (500t - 89^\circ\ 46').$$

## CHAPTER V

# PROBLEMS INVOLVING DISTRIBUTED CAPACITY AND INDUCTANCE[1]

**47.** It was pointed out in Par. 18 that, while the solution of ordinary network problems would, in general, lead to operational expressions of a comparatively simple character, the application of the same methods to problems involving distributed capacities and inductances would give rise to circular, hyperbolic, and exponential functions of operators, the interpretations of which have not yet been considered.

It is proposed in this chapter to illustrate the application of the vectorial method to problems of this type, by discussing briefly the theory of the action of the telegraph or telephone cable, with some account of the production of stationary oscillations on wires. For this purpose it will first be necessary to consider some of the types of operational expressions which are likely to occur in these and similar analyses.

**48. Circular and Hyperbolic Functions of Operators.**—It was shown in Par. 12 that the operator $(\cos\theta + \mathrm{j}\sin\theta)$ could be expressed in the form $\epsilon^{\mathrm{j}\theta}$, this being a short way of writing the series

$$1 + \mathrm{j}\theta + \frac{(\mathrm{j}\theta)^2}{2!} + \frac{(\mathrm{j}\theta)^3}{3!} + \frac{(\mathrm{j}\theta)^4}{4!} + \text{etc., etc. } \textit{ad inf.} \qquad (461)$$

It was further shown that though, from a rigidly mathematical point of view, $\epsilon^{\mathrm{j}\theta}$ could only be regarded as a short way of writing the series, yet the $\mathrm{j}\theta$ does really appear to be of the nature of an index, since it can be shown to obey the ordinary index laws. On this understanding, therefore,

$$(\cos\theta + \mathrm{j}\sin\theta) = \epsilon^{\mathrm{j}\theta}. \qquad (462)$$

Similarly, putting $-\theta$ for $\theta$,

$$(\cos\theta - \mathrm{j}\sin\theta) = \epsilon^{-\mathrm{j}\theta}. \qquad (463)$$

[1] Bibliography, Nos. 10, 13, 6, 3, 1.

By the addition and subtraction of these two equation

$$\cos\theta = \frac{(\epsilon^{j\theta} + \epsilon^{-j\theta})}{2}. \tag{46-}$$

$$\sin\theta = \frac{1}{j}\frac{(\epsilon^{j\theta} - \epsilon^{-j\theta})}{2}. \tag{46.}$$

In considering operational expressions, such as sin jb, cos j etc., the above process can be, as it were, reversed and the exp nential expressions on the right-hand sides of Eqs. (464) and (46 considered as defining, in conjunction with the series of Eq. (461 the circular functions on the left-hand sides of the equatior This procedure will solve the difficulty of the interpretation circular functions of operators, though, of course, it remains be proved that such functions, defined in this way, obey t] ordinary formulas of trigonometry.[1]

The method of proving that the formulas of trigonometry d in fact, apply to such functions can be illustrated by two typic cases. For instance,

$$\text{Sin a} \cos jb + \cos a \sin jb. \tag{46}$$

$$= \frac{1}{j}\frac{(\epsilon^{ja} - \epsilon^{-ja})(\epsilon^{-b} + \epsilon^{b})}{4} + \frac{1}{j}\frac{(\epsilon^{ja} + \epsilon^{-ja})(\epsilon^{-b} - \epsilon^{b})}{4} \tag{46}$$

$$= \frac{1}{4j}(2\epsilon^{ja}\epsilon^{-b} - 2\epsilon^{-ja}\epsilon^{b})$$

$$= \frac{1}{2j}\left\{\epsilon^{j(a+jb)} - \epsilon^{-j(a+jb)}\right\} \tag{46}$$

$$= \sin(a + jb). \tag{46}$$

Therefore, $\sin(a + jb) = (\sin a \cos jb + \cos a \sin jb)$. (47

or again $\quad 2\sin jb \cos jb = \frac{2}{j}\frac{(\epsilon^{-b} - \epsilon^{b})}{2}\frac{(\epsilon^{-b} + \epsilon^{b})}{2}$ (47

$$= \frac{2}{j}\frac{(\epsilon^{-2b} - \epsilon^{2b})}{4} \tag{47}$$

$$= \frac{1}{j}\frac{(\epsilon^{j(2jb)} - \epsilon^{-j(2jb)})}{2} \tag{47}$$

$$= \sin 2jb. \tag{47}$$

Considering now the hyperbolic functions, which are defin thus

$$\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \ . \ . \ . + \textit{etc., etc. ad. inf.} \tag{47}$$

$$= \frac{\epsilon^{u} - \epsilon^{-u}}{2} \tag{47}$$

[1] Bibliography, No. 23.

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \quad . \quad . \quad . \quad + \textit{etc., etc. ad. inf.} \qquad (477)$$

$$= \frac{\epsilon^u + \epsilon^{-u}}{2}, \qquad (478)$$

then sinh jb is seen to be the operator $\frac{\epsilon^{jb} - \epsilon^{-jb}}{2}$, and cosh jb the operator $\frac{\epsilon^{jb} + \epsilon^{-jb}}{2}$ in which, as before, $\epsilon^{jb}$ is to be regarded as an abbreviation for the series Eq. (461). On the basis of this definition, and since the quantities such as jb obey the index laws, it is easily shown exactly as in the corresponding cases of the circular functions that *hyperbolic functions of operators obey the ordinary formulas of hyperbolic trigonometry.*

Further, on the basis of the exponential definitions of both the hyperbolic and the circular functions,

$$\sin jb = \frac{1}{j}\,\frac{\epsilon^{jjb} - \epsilon^{-jjb}}{2} \qquad (479)$$

$$= \frac{1}{j}\,\frac{\epsilon^{-b} - \epsilon^{b}}{2} \qquad (480)$$

$$= j\,\frac{\epsilon^{b} - \epsilon^{-b}}{2} \qquad (481)$$

$$= j \sinh b. \qquad (482)$$

Similarly,

$$\cos jb = \frac{\epsilon^{jjb} + \epsilon^{-jjb}}{2} \qquad (483)$$

$$= \frac{\epsilon^{b} + \epsilon^{-b}}{2} \qquad (484)$$

$$= \cosh b. \qquad (485)$$

In this way the equalities given in the following table may be derived:

$$\left.\begin{array}{ll} \sin jb = j \sinh b & \sinh jb = j \sin b \\ \cos jb = \ \ \cosh b & \cosh jb = \ \ \cos b \\ \tan jb = j \tanh b & \tanh jb = j \tan b \end{array}\right\}. \qquad (486)$$

The reduction to one or the other of the standard forms of any of the circular or the hyperbolic functions which are likely to occur in practice may now be considered. For instance,

$$\sin (a + jb) = \sin a \cos jb + \cos a \sin jb \qquad (487)$$

$$= (\sin a \cosh b) + j(\cos a \sinh b), \qquad (488)$$

and, again,

$$\sinh (a + jb) = \sinh a \cosh jb + \cosh a \sinh jb \qquad (489)$$

$$= (\sinh a \cos b) + j\,(\cosh a \sin b). \qquad (490)$$

For tan (a + jb),

$$\tan (a + jb) = \frac{\sin a \cosh b + j \cos a \sinh b}{\cos a \cosh b + j \sin a \sinh b}. \qquad (491)$$

This expression could be further simplified by conversion into the form $r\epsilon^{j\theta}$ by the methods already described, but in this case it is simpler to operate on the numerator and denominator with ($\cos a \cosh b + j \sin a \sinh b$), giving

$$\frac{(\cosh^2 b - \sinh^2 b) \sin a \cos a + j (\sin^2 a + \cos^2 a) \sinh a \cosh b}{\cos^2 a \cosh^2 b + \sin^2 a \sinh^2 b} \qquad (492)$$

which, by means of the usual trigonometrical formulas, can be reduced to

$$\frac{\sin 2a + j \sinh 2b}{\cos 2a + \cosh 2b}. \qquad (493)$$

The above examples will suffice to illustrate the method. For convenience of reference, a list of the more usual forms is given in Par. 50.

**49. Exponential and Logarithmic Functions of Operators.**—
1 The form $\epsilon^{(a+jb)}$ presents no difficulty, since

$$\epsilon^{(a+jb)} = \epsilon^{a}\epsilon^{jb}. \qquad (494)$$

2. For log (a + jb) the result is, putting

$$(a + jb) = r\epsilon^{j\theta} \qquad (495)$$

$$\log (a + jb) = \log r\epsilon^{j\theta} \qquad (496)$$

$$= \log r + \log \epsilon^{j\theta} \qquad (497)$$

$$= \log r + j\theta \qquad (498)$$

$$= \log (a^2 + b^2)^{\frac{1}{2}} + j \tan^{-1} \frac{b}{a}. \qquad (499)$$

**50. Standard Forms.**—The standard forms tabulated below for convenience of reference can all be established by the methods described in the preceding paragraphs.

$$\tan (a + jb) = \frac{\sin 2a + j \sinh 2b}{\cos 2a + \cosh 2b}. \qquad (500)$$

$$\cot (a + jb) = \frac{\sin 2a - j \sinh 2b}{\cosh 2b - \cos 2a}. \qquad (501)$$

$$\operatorname{cosec} (a + jb) = 2\, \frac{\sin a \cosh b - j \cos a \sinh b}{\cosh 2b - \cos 2a}. \qquad (502)$$

$$\sec (a + jb) = 2\, \frac{\cos a \cosh b + j \sin a \sinh b}{\cos 2a + \cosh 2b}. \qquad (503)$$

$$\tanh (a + jb) = \frac{\sinh 2a + j \sin 2b}{\cosh 2a + \cos 2b} \qquad (504)$$

$$\operatorname{cotanh}(a + jb) = \frac{\sinh 2a - j \sin 2b}{\cosh 2a - \cos 2b}. \tag{505}$$

$$\operatorname{cosech}(a + jb) = 2\,\frac{\cos b \sinh a - j \cosh a \sin b}{\cosh 2a - \cos 2b}. \tag{506}$$

$$\operatorname{sech}(a + jb) = 2\,\frac{\cosh a \cos b - j \sinh a \sin b}{\cosh 2a + \cos 2b}. \tag{507}$$

$$\log \sin(a + jb) = \frac{1}{2}\log\frac{\cosh 2b - \cos 2a}{2} + j \tan^{-1}(\cot a \tanh b). \tag{508}$$

$$\log \cos(a + jb) = \frac{1}{2}\log\frac{\cosh 2b + \cos 2a}{2} - j \tan^{-1}(\tan a \tanh b). \tag{509}$$

**51. Current and Potential Distribution in Telephone and Telegraph Cables.**—Consider the arrangement shown in Fig. 49, which represents a pair of conductors of indefinite length,

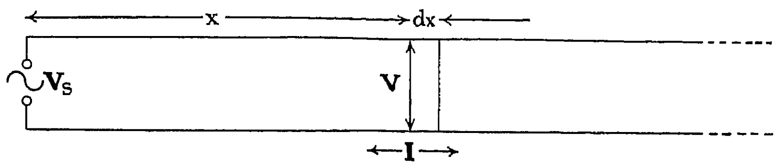


FIG. 49.

between the ends of which is applied an e.m.f. $v_s = \hat{V}_s \cos \omega t$. Let L, C, R, and G be, respectively, the inductance, the capacity, the resistance, and the leakance or dielectric conductance per unit length of the pair of lines. Then it is easily shown[1] that the differential equations for the current in the lines and the potential difference between them at a point a distance X from the sending end, *i.e.*, the end at which the e.m.f. $v_s$ is applied, take the form

$$-\frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t} \tag{510}$$

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t}. \tag{511}$$

For the reasons given in Par. 31, v and i may be assumed to be simple harmonic functions of time of the same frequency as $v_s$. Therefore

$$i = \mathbf{I} \cdot \boldsymbol{\nu} \tag{512}$$

$$v = \mathbf{V} \cdot \boldsymbol{\nu} \tag{513}$$

[1] See "The Propagation of Electric Currents in Telephone and Telegraph Conductors," by J. A. Fleming.

and transforming the scalar Eqs. (510) and (511) into the vector form

$$-\frac{\partial}{\partial x}(\mathbf{I}\cdot\boldsymbol{\nu}) = G(\mathbf{V}\cdot\boldsymbol{\nu}) + C\frac{\partial}{\partial t}(\mathbf{V}\cdot\boldsymbol{\nu}) \tag{514}$$

$$-\frac{\partial}{\partial x}(\mathbf{V}\cdot\boldsymbol{\nu}) = R(\mathbf{I}\cdot\boldsymbol{\nu}) + L\frac{\partial}{\partial t}(\mathbf{I}\cdot\boldsymbol{\nu}), \tag{515}$$

*i.e.*

$$\frac{\partial \mathbf{I}}{\partial x}\cdot\boldsymbol{\nu} = (G + j\omega C)\mathbf{V}\cdot\boldsymbol{\nu} \tag{516}$$

$$\frac{\partial \mathbf{V}}{\partial x}\cdot\boldsymbol{\nu} = (R + j\omega L)\mathbf{I}\cdot\boldsymbol{\nu} \tag{517}$$

and, since these equations hold for all values of t,

$$-\frac{\partial \mathbf{I}}{\partial x} = (G + j\omega C)\mathbf{V} \tag{518}$$

$$-\frac{\partial \mathbf{V}}{\partial x} = (R + j\omega L)\mathbf{I}. \tag{519}$$

Differentiating with respect to x,

$$-\frac{\partial^2 \mathbf{I}}{\partial x^2} = (G + j\omega C)\frac{\partial \mathbf{V}}{\partial x} = -(G + j\omega C)(R + j\omega L)\mathbf{I} \tag{520}$$

$$-\frac{\partial^2 \mathbf{V}}{\partial x^2} = (R + j\omega L)\frac{\partial \mathbf{I}}{\partial x} = -(R + j\omega L)(G + j\omega C)\mathbf{V}. \tag{521}$$

Thus, putting

$$P^2 = (G + j\omega c)(R + j\omega L) \tag{522}$$

the result is

$$\frac{\partial^2 \mathbf{I}}{\partial x^2} = P^2\mathbf{I} \tag{523}$$

$$\frac{\partial^2 \mathbf{V}}{\partial x^2} = P^2\mathbf{V}. \tag{524}$$

Although these are vector equations, the differential coefficients having the extended meaning of vector differentiation, it can easily be shown that their solutions are of the same form as would satisfy the same equations with scalar variables. Thus, putting

$$\mathbf{V} = \epsilon^{Px}\mathbf{A} + \epsilon^{-Px}\mathbf{B}, \tag{525}$$

where **A** and **B** are arbitrary constant vectors, the result is

$$\frac{\partial \mathbf{V}}{\partial x} = P(\epsilon^{Px}\mathbf{A} - \epsilon^{-Px}\mathbf{B}) \tag{526}$$

and

$$\frac{\partial^2 \mathbf{V}}{\partial x^2} = P^2(\epsilon^{Px}\mathbf{A} + \epsilon^{-Px}\mathbf{B}) \tag{527}$$

$$= P^2\mathbf{V}. \tag{528}$$

Equation (525), therefore, satisfies Eq. (524) and, since it has the requisite number of arbitrary constants, it can be regarded as the general solution of Eq. (524).

The values of the arbitrary constant vectors **A** and **B** will, of course, depend on the particular case being considered. A full development of the solution, such as is given in the book to which reference has been made, is outside the scope of the present work, but, as illustrations of the method to be adopted, the special cases of a line of length l short-circuited at the distant or receiving end, and the simpler case of a line so long as to be virtually of infinite length, will be considered.

In each case

$$\mathbf{V} = \mathbf{V}_s, \text{ when } x = 0 \tag{529}$$

$$\therefore \mathbf{A} + \mathbf{B} = \mathbf{V}_s. \tag{530}$$

Further, for the line of length l short-circuited at the receiving end

$$\mathbf{V} = 0 \text{ when } x = l, \tag{531}$$

so that

$$\epsilon^{Pl}\mathbf{A} + \epsilon^{-Pl}\mathbf{B} = 0 \tag{532}$$

$$\therefore \quad \mathbf{A} = \frac{1}{1 - \epsilon^{2Pl}} \mathbf{V}_s \tag{533}$$

$$= \frac{\epsilon^{-Pl}}{\epsilon^{-Pl} - \epsilon^{Pl}} \mathbf{V}_s \tag{534}$$

$$= -\frac{\epsilon^{-Pl}}{2 \sinh Pl} \mathbf{V}_s \tag{535}$$

and, similarly,

$$\mathbf{B} = \frac{\epsilon^{Pl}}{2 \sinh Pl} \mathbf{V}_s \tag{536}$$

$$\therefore \quad \mathbf{V} = \frac{1}{2 \sinh Pl} \{ -\epsilon^{Px}\epsilon^{-Pl} + \epsilon^{-Px}\epsilon^{Pl} \} \mathbf{V}_s \tag{537}$$

$$= \frac{1}{2 \sinh Pl} \{ \epsilon^{P(l-x)} - \epsilon^{-P(l-x)} \} \mathbf{V}_s \tag{538}$$

$$= \frac{\sinh P(l - x)}{\sinh Pl} \mathbf{V}_s. \tag{539}$$

Also, since

$$\mathbf{I} = -\frac{1}{R + j\omega L} \frac{\partial \mathbf{V}}{\partial x}, \tag{540}$$

$$\mathbf{I} = \frac{P}{R + j\omega L} \frac{\cosh P(l - x)}{\sinh Pl} \mathbf{V}_s \tag{541}$$

Equations (539) and (541) can be put in the form

$$\mathbf{V} = (\cosh Px - \coth Pl \sinh Px)\, \mathbf{V}_s \tag{542}$$

$$\mathbf{I} = \frac{P}{R + j\omega L}(\coth Pl \cosh Px - \sinh Px)\mathbf{V}_s. \tag{543}$$

This is the complete vectorial solution for **V** and **I** in terms of $\mathbf{V}_s$ and x. For the physical interpretation of the result, it will be necessary to consider more in detail the operational expres-

sions involved and to transform the hyperbolic functions into exponentials.

For P,

$$P^2 = (G + j\omega c)(R + j\omega L). \tag{544}$$

Putting

$$(R + j\omega L) = r_1\epsilon^{j\theta_1} \tag{545}$$

$$(G + j\omega c) = r_2\epsilon^{j\theta_2}, \tag{546}$$

we have

$$P = \sqrt{r_1r_2}\,\epsilon^{\frac{j(\theta_1+\theta_2)}{2}} \tag{547}$$

and

$$\frac{P}{R + j\omega L} = \sqrt{\frac{r_1}{r_2}}\,\epsilon^{\frac{j(\theta_1-\theta_2)}{2}} \tag{548}$$

$$\therefore\ P = (a + jb), \tag{549}$$

where

$$a = \sqrt{r_1r_2}\cos\frac{(\theta_1 + \theta_2)}{2} \tag{550}$$

$$= \frac{1}{\sqrt{2}}\sqrt{r_1r_2 + (RG - \omega^2LC)} \tag{551}$$

and

$$b = \sqrt{r_1r_2}\sin\frac{(\theta_1 + \theta_2)}{2} \tag{552}$$

$$= \frac{1}{\sqrt{2}}\sqrt{r_1r_2 - (RG - \omega^2LC)}. \tag{553}$$

As shown in Par. 50,

$$\frac{1}{\sinh Pl} = \operatorname{cosech} Pl \tag{554}$$

$$= h + jk, \tag{555}$$

where

$$h = 2\,\frac{\cos bl \sinh al}{\cosh 2al - \cos 2bl} \tag{556}$$

and

$$k = -2\,\frac{\cosh al \sin bl}{\cosh 2al - \cos 2bl}. \tag{557}$$

Thus,

$$\frac{1}{\sinh Pl} = r_3\epsilon^{j\theta_3}, \tag{558}$$

where

$$r_3 = \sqrt{h^2 + k^2} \tag{559}$$

and

$$\tan\theta_3 = \frac{k}{h}. \tag{560}$$

The operator $\frac{P}{(R + j\omega L)\sinh Pl}$, therefore, which appears in Eq. (541), becomes

$$r_3\sqrt{\frac{r_1}{r_2}}\,\epsilon^{j\left(\theta_3+\frac{\theta_1}{2}-\frac{\theta_2}{2}\right)}, \tag{561}$$

which for compactness may be written $r\epsilon^{j\theta}$,

where

$$r = r_3\sqrt{\frac{r_1}{r_2}} \tag{562}$$

and

$$\theta = \theta_3 + \frac{\theta_1}{2} - \frac{\theta_2}{2}, \tag{563}$$

so that Eq. (541) may be written

$$I = r\epsilon^{j\theta} \cosh P(l - x)\mathbf{V}_s \tag{564}$$

$$= \frac{r\epsilon^{j\theta}}{2}\left\{\epsilon^{a(l-x)}\,\epsilon^{jb(l-x)} + \epsilon^{-a(l-x)}\,\epsilon^{-jb(l-x)}\right\}\mathbf{V}_s \tag{565}$$

and, since

$$\mathbf{V}_s \cdot \boldsymbol{\nu} = \hat{V}_s \cos \omega t, \tag{566}$$

the result for i, *i.e.*, for $\mathbf{I} \cdot \boldsymbol{\nu}$, is

$$i = \frac{r}{2}\epsilon^{al}\,\epsilon^{-ax}\,\hat{V}_s \cos(\omega t + \theta + bl - bx) \tag{567}$$

$$+ \frac{r}{2}\epsilon^{-al}\,\epsilon^{ax}\,\hat{V}_s \cos(\omega t + \theta - bl + bx). \tag{568}$$

Thus, the scalar form of the solution is essentially

$$i = k_1\,\epsilon^{-ax}\,\hat{V}_s \cos(\omega t - bx + \psi_1) + k_2\epsilon^{ax}\,\hat{V}_s \cos(\omega t + bx + \psi_2), \tag{569}$$

$k_1$, $k_2$, $\psi_1$, and $\psi_2$ being constants dependent on the characteristics of the line and on its length.

*The expression for the current is seen to be composed of two terms, each of which represents a progressive wave. The first of these is traveling in the positive direction of* x, *and its amplitude decreases exponentially in this direction. The second is traveling in the reverse direction, and its amplitude decreases exponentially in the direction in which it is traveling.*

From a practical point of view, a and b, the coefficients of x, appear to be the important features of the line. Their values in terms of the electrical constants of the line have already been determined (Eqs. (550) and (553)).

*The first,* a, *is known as the attenuation factor of the line, since it is the measure of the exponential decrement of the amplitude of the wave.*

*The second,* b, *is known as the wave-length constant, since the wave length of the progressive waves is* $\frac{2\pi}{b}$.

Since the frequency f is $\frac{\omega}{2\pi}$, for the velocity of the waves

$$v = f \times \text{wave length} \tag{570}$$

$$= \frac{\omega}{2\pi}\cdot\frac{2\pi}{b} = \frac{\omega}{b}. \tag{571}$$

The above is, of course, only an outline of the physical interpretation of the equations. More space, however, will not be

devoted to it, as the present book is only concerned with the mathematical side of the subject.

**52. The Special Case of a Very Long Line.**—In many practical cases l is so great that it can be considered to be virtually infinite. This approximation can be made to a high degree of accuracy in all cases in which the product al is greater than 4. It will be seen on reference to a table of hyperbolic functions that, when al is greater than 4, cosh al is practically equal to sinh al, so that

$$\coth \mathrm{Pl} = \frac{\cosh (\mathrm{al} + \mathrm{jbl})}{\sinh (\mathrm{al} + \mathrm{jbl})} \tag{572}$$

$$= \frac{\cosh \mathrm{al} \cos \mathrm{bl} + \mathrm{j} \sinh \mathrm{al} \sin \mathrm{bl}}{\sinh \mathrm{al} \cos \mathrm{bl} + \mathrm{j} \cosh \mathrm{al} \sin \mathrm{bl}} \tag{573}$$

$$= 1. \tag{574}$$

Under these conditions, therefore, Eq. (543) becomes

$$\mathrm{I} = \frac{\mathrm{P}}{\mathrm{R} + \mathrm{J}\omega\mathrm{L}} (\cosh \mathrm{Px} - \sinh \mathrm{Px})\, \mathrm{V_s} \tag{575}$$

$$= \frac{\mathrm{P}}{\mathrm{R} + \mathrm{j}\omega\mathrm{L}} \epsilon^{-\mathrm{Px}} \mathrm{V_s} = \left(\frac{\mathrm{G} + \mathrm{j}\omega\mathrm{C}}{\mathrm{R} + \mathrm{j}\omega\mathrm{L}}\right)^{\frac{1}{2}} \epsilon^{-\mathrm{Px}} \mathrm{V_s}. \tag{576}$$

$$\tag{577}$$

*The quantity* $\left\{\frac{(\mathrm{G} + \mathrm{J}\omega\mathrm{C})}{(\mathrm{R} + \mathrm{J}\omega\mathrm{L})}\right\}^{-\frac{1}{2}}$ *is known as the "vector characteristic impedance" of the line.* It is obviously the ratio of the vectors representing the potential and the current at the sending end of a line of infinite length. Expressing it in the form $\mathrm{z_c}\epsilon^{\mathrm{j}\psi}$, where

$$\mathrm{z_c} = \left\{\frac{(\mathrm{R}^2 + \omega^2\mathrm{L}^2)}{(\mathrm{G}^2 + \omega^2\mathrm{C}^2)}\right\}^{\frac{1}{4}}; \tag{578}$$

$$\psi = \tfrac{1}{2} \tan^{-1} \frac{\omega\mathrm{L}}{\mathrm{R}} - \tfrac{1}{2} \tan^{-1} \frac{\omega\mathrm{C}}{\mathrm{G}}. \tag{579}$$

Equation (577) can be written

$$\mathrm{I} = \frac{1}{\mathrm{z_c}} \epsilon^{-\mathrm{ax}} \epsilon^{-\mathrm{j}(\mathrm{bx} + \psi)} \mathrm{V_s}, \tag{580}$$

*i.e.*,

$$\mathrm{i} = \frac{1}{\mathrm{z_c}} \epsilon^{-\mathrm{ax}} \hat{\mathrm{V}}_{\mathrm{s}} \cos (\omega \mathrm{t} - \mathrm{bx} - \psi). \tag{581}$$

Comparing this with Eq. (569) it is seen that *in a line of very great or infinite length there is no reverse direction or reflected wave.*

**53. Stationary Waves on Wires.**—The discussion of the formation of stationary oscillations on wires can be regarded

as the special case of the preceding analysis which arises when R and G are so small as to be negligible compared with $\omega L$ and $\omega C$ respectively. At radio frequencies these conditions are generally fulfilled, since $\omega$ is then very great. The discussion of this special case will only differ from that of Par. 31 in the value of P. Putting $G = R = 0$ in

$$P^2 = (G + j\omega C)(R + j\omega L), \tag{582}$$

the result is

$$P^2 = -\omega^2 LC, \tag{583}$$

so that

$$P = \frac{j\omega}{p}, \tag{584}$$

where

$$p^2 = \frac{1}{LC}. \tag{585}$$

Applying this in the solution of the case in which

$$V = V_s \quad \text{at } x = 0 \tag{586}$$

$$V = 0 \quad \text{at } x = l, \tag{587}$$

*i.e.*, in the case of a pair of lines of length l short-circuited at the far end, the result is, as in Eq. (539),

$$V = \frac{\sinh \frac{j\omega}{p}(l - x)}{\sinh \frac{j\omega}{p} l} V_s \tag{588}$$

$$I = \frac{\frac{j\omega}{p}}{j\omega L} \frac{\cosh \frac{j\omega}{p}(l - x)}{\sinh \frac{j\omega}{p} l} V_s \tag{589}$$

*i.e.*,

$$V = \frac{\sin \frac{\omega}{p}(l - r)}{\sin \frac{\omega l}{p}} V_s \tag{590}$$

and

$$I = \frac{1}{jpL} \frac{\cos \frac{\omega}{p}(l - x)}{\sin \frac{\omega l}{p}} V_s \tag{591}$$

The equations take a somewhat simpler form if x be measured from the far end instead of from the point of application of the e.m.f. Then

$$V = \frac{\sin \frac{\omega x}{p}}{\sin \frac{\omega l}{p}} V_s \tag{592}$$

and
$$I = \frac{1}{jpL}\,\frac{\cos\frac{\omega x}{p}}{\sin\frac{\omega l}{p}}\,V_s. \tag{593}$$

In scalar form these become, since $V_s \cdot \mathbf{v} = \hat{V}_s \cos \omega t$

$$v = \frac{\hat{V}_s}{\sin\frac{\omega l}{p}} \sin\frac{\omega x}{p} \cos \omega t \tag{594}$$

$$i = \frac{\hat{V}_s}{Lp \sin\frac{\omega l}{p}} \cos\frac{\omega x}{p} \sin \omega t. \tag{595}$$

Thus, the distribution of current and of potential is in the form of a standing wave with a node of potential at $x = 0$, *i.e.*, at the short-circuited end, and an antinode of current at the same point.

It is clear that nodes of potential will occur when

$$\sin\frac{\omega x}{p} = 0, \tag{596}$$

*i.e.*,
$$\frac{\omega x}{p} = 0 \text{ or } n\pi \tag{597}$$

or
$$x = \frac{n\pi p}{\omega}. \tag{598}$$

Similarly, the amplitude of v will become a maximum (theoretically infinite) whenever

$$\sin\frac{\omega l}{p} = 0, \tag{599}$$

*i.e.*,
$$\frac{\omega l}{p} = 0 \text{ or } n\pi, \tag{600}$$

*i.e.*,
$$l = \frac{pn\pi}{\omega}. \tag{601}$$

In the above, p has been written for $\frac{1}{\sqrt{LC}}$. For two parallel wires of diameter d in free space, with centers D centimeters apart[1]

[1] This is true when the linkages within the wires are negligibly small, *i.e.*, when $\frac{d}{D}$ is small or when $\omega$ is so high that the "skin effect" confines the current to the periphery of the wire.

$$L = 4 \log_\epsilon \frac{2D}{d} \qquad \text{E.M. units per centimeter} \quad (602)$$

$$C = \frac{1}{4 \log_\epsilon \frac{2D}{d}} \qquad \text{E.S. units per centimeter} \quad (603)$$

$$= \frac{1}{4 \log_\epsilon \frac{2D}{d}} \cdot \frac{1}{c^2} \qquad \text{E.M. units per centimeter} \quad (604)$$

where $c = 3 \times 10^{10}$ cm. per second.
*i.e.*, c is equal to the velocity of light in free space. Therefore,

$$LC = \frac{1}{c^2} = \text{const} \qquad (605)$$

In the above equations, therefore, p is an absolute constant for all values of d and D, provided $\frac{d}{D}$ is small and is, in fact, the velocity of light in free space.

**EXAMPLES**

**1.** A long telephone line has the following constants per loop-mile, *i.e.*, per mile of lead and return:

| | |
|---|---|
| Resistance | 100 ohms. |
| Inductance | 1 millihenry. |
| Capacity | .05 microfarads. |
| Leakance | nil. |

A sinusoidal e.m.f. the amplitude of which is 0.1 volt, the frequency being 159 p.p.s., is applied at one end of the line.
The e.m.f. being represented by the vector .1v, find the corresponding vector expressions for the current in the line at the following distances from the end at which the e.m.f. is applied:

(*a*) 0 miles. (*b*) 62.5 miles. (*c*) 125 miles.

**2.** Assuming that the primary constants of the above line are independent of frequency, find (*a*) the attenuation factor, (*b*) the wave-length constant, (*c*) the wave length, and (*d*) the wave velocity of the line at the frequencies (1) 1,590 p.p.s. and (2) 159,000 p.p.s.

**3.** Under what conditions will both the attenuation factor and the wave velocity of a telephone line be independent of frequency?

**4.** A cable having the following constants:

| | |
|---|---|
| Inductance per unit loop length | L |
| Resistance per unit loop length | R |
| Capacity per unit loop length | C |
| Leakance per unit loop length | G |
| Loop length | 1 |
| Receiving-end impedance | $z_r = R_r + jX_r$ |

is supplied with current at the sending end by an e.m.f. of frequency $\frac{\omega}{2\pi}$ represented by the vector $V_S$.

Find the vector expression for the current at a point distant x from the sending end.

For what value of $z_r$ will this expression be the same as a line of infinite length having the same primary constants?

**5.** (*a*) A uniform helix of copper wire has the following constants:

| | |
|---|---|
| Length .................. | 5 meters |
| Inductance................. | .149 millihenries per unit length. |
| Capacity.................. | .21 micro-microfarads per unit length. |
| Resistance................ | negligible. |

An e.m.f. of amplitude 10 volts and frequency 250,000 p.p.s. is applied between one end and earth.

Write the expression for the current at a point distant x from the open end, taking the origin of time as the instant when the e.m.f. is a maximum.

Give the vector expression for the current for the following values of x:

(*b*) 0.

(*c*) 1.785 meters.

(*d*) 3.57 meters.

## ANSWERS TO EXAMPLES

**1.** (*a*) $70.3 \times 10^{-6}\epsilon^{j44°43'}$v.
(*b*) $-3.16 \times 10^{-6}\epsilon^{j44°43'}$v.
(*c*) $.1406 \times 10^{-6}\epsilon^{j44°43'}$v.

**2.** (1) (*a*) .1505.
(*b*) .166.
(*c*) 37.8 miles.
(*d*) 60,250 miles per second.
(2) (*a*) .354.
(*b*) 7.08.
(*c*) .8860 miles.
(*d*) 141,000 miles per second.

**3.** If the product of the inductance and the leakance is equal to the product of the capacity and the resistance (each per unit length), then the attenuation factor and the wave velocity will both be independent of frequency, *i.e.*, if

$$\mathrm{LG} = \mathrm{CR}$$
$$\text{Attenuation factor} = \sqrt{\mathrm{GR}}$$
$$\text{Wave velocity} = \frac{1}{\sqrt{\mathrm{LC}}}.$$

If the leakance (G) is nil, and if $\omega$L is very large compared with R, then:

$$\text{Attenuation factor} = \frac{\mathrm{R}}{2}\sqrt{\frac{\mathrm{C}}{\mathrm{L}}}$$
$$\text{Wave velocity} = \frac{1}{\sqrt{\mathrm{LC}}}.$$

**4.** $I = \left(\frac{1}{z_0}\cosh Px - \frac{1}{z_c}\sinh Px\right)V_s,$

where $$z_0 = z_c\left(\frac{z_r \cosh Pl + z_c \sinh Pl}{z_c \cosh Pl + z_r \sinh Pl}\right)$$

and $$z_c = \sqrt{\frac{R + j\omega L}{G + j\omega C}}.$$

The current will be the same as in a line of infinite length if $z_r = z_c$.

**5.** (*a*) $I = \frac{\sin 8.8 \times 10^{-3}\,x}{\cos 4.4}\, 3.75 \times 10^{-5} \times j \times 10v$

$= (-1.22\ 10^{-3}\ j \sin .0088x)v.$

(*b*) 0.

(*c*) $-1.22\ 10^{-3} jv.$

(*d*) 0.

## CHAPTER VI

## DAMPED ELECTRIC OSCILLATIONS

**54. Vectorial Representation of a Damped Electric Oscillation.**—Consider an alternating current whose instantaneous value i is given by the equation

$$\mathbf{i} = \hat{I}_0\epsilon^{-kt} \cos (\omega t + \psi). \tag{606}$$

It has been seen that such a quantity can be represented by a vector **I** of magnitude

$$\hat{I} = \hat{I}_0\epsilon^{-kt} \tag{607}$$

rotating with uniform angular velocity $\omega$. It has also been shown that for a vector of this description

$$\frac{dI}{dt} = (\omega j - k)\mathbf{I} \tag{608}$$

and

$$\frac{d^2I}{dt^2} = (\omega j - k)^2 I \text{ etc., etc.} \tag{609}$$

An alternating current of this type is known as a damped electric oscillation.

**55. The Application of Kirchhoff's Laws to Vectors Representing Damped Electric Oscillations.**—It was shown in Par. 34 that Kirchhoff's first and second laws could be applied to vectors representing alternating potentials and currents of sine-wave form. The proof given required only that the scalar products of the vectors concerned with the fixed unit vector correctly represented the instantaneous values of the electric quantities, and will, therefore, apply equally to vectors of the type under consideration, *i.e.*, to vectors representing damped electric oscillations.

**56. Current and Potential Relationships for Damped Electric Oscillations.**—As shown in Par. 35, there are for $\mathbf{E}_R$, $\mathbf{E}_L$, and $\mathbf{E}_C$, the vectors which represent the potential differences due to the passage of a current represented by the vector **I** through a resistance R, an inductance L, and a capacity C respectively, the equations

$$\mathbf{E}_R = -RI \tag{610}$$

$$\mathbf{E}_L = -L\frac{dI}{dt} \tag{611}$$

ınd
$$\frac{dE_C}{dt} = -\frac{I}{C}. \tag{612}$$

It has also been shown (see Par. 28) that, for vectors of the type ınder consideration,
$$\frac{dI}{dt} = (\omega j - k)\mathbf{I}. \tag{613}$$

Therefore, for the damped electric oscillation represented by he vector **I**,
$$\mathbf{E}_R = -R\mathbf{I} \tag{614}$$
$$\mathbf{E}_L = -L(\omega j - k)\mathbf{I} \tag{615}$$
$$(\omega j - k)\mathbf{E}_C = -\frac{I}{C} \tag{616}$$
ır
$$\mathbf{E}_C = -\frac{1}{C(\omega j - k)}\,\mathbf{I} = \frac{\omega j + k}{Cp^2}\,\mathbf{I}. \tag{617}$$

Where $p^2 = \omega^2 + k^2$. (618)

The vectors $\mathbf{E}_R$, $\mathbf{E}_L$, $\mathbf{E}_C$ are illustrated in Figs. 50, 51, and 52 espectively. For the reasons given in Par. 35, they will have he same angular velocity and the same decrement of magnitude ,s the vector **I**.

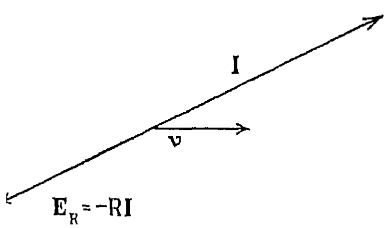


Fig. 50.

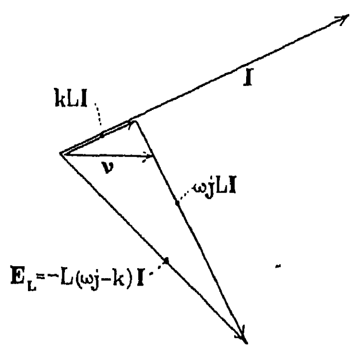


Fig. 51.

**57. Energy Conditions in Damped Electric Oscillations.**—One nportant difference between damped and undamped electric scillations is immediately apparent in Figs. 51 and 52, namely, hat the current and the potential vectors in the cases of a pure ıductance and a pure capacity are not 90° apart as they were in he corresponding cases with undamped oscillations. It follows 'om this that the mean rate of change of energy associated with onductors of this type is no longer zero.

Consider, for instance, the case of an inductance L through which is passing an electric oscillation represented by the vector I, where

$$\mathbf{I} \cdot \nu = \hat{I}_0 \epsilon^{-kt} \cos \omega t. \tag{619}$$

As shown in Eq. (615),

$$\mathbf{E}_L = -L(\omega j - k)\mathbf{I} \tag{620}$$

$$= L(\omega^2 + k^2)^{\frac{1}{2}} \epsilon^{j\psi} \mathbf{I}, \tag{621}$$

where
$$\tan \psi = -\frac{\omega}{k}. \tag{622}$$

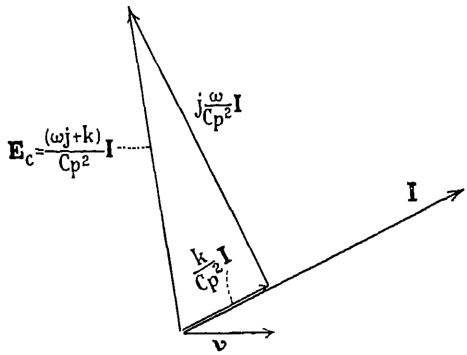

FIG. 52.

Applying the energy equation (Eq. (325)), we have for $p_L$, the instantaneous value of the rate of change of the energy associated with L,

$$p_L = \frac{\mathbf{I} \cdot \mathbf{E}_L}{2} + \mathbf{W} \cdot \mathbf{v} \tag{623}$$

$$= -\frac{\mathbf{I} \cdot L(\omega j - k)\mathbf{I}}{2} + \frac{L(\omega^2 + k^2)^{\frac{1}{2}} \hat{I}^2}{2} \cos (2\omega t + \psi) \tag{624}$$

$$= \frac{kL\hat{I}^2}{2} + \left\{\frac{L(\omega^2 + k^2)^{\frac{1}{2}} \hat{I}^2}{2}\right\} \cos (2\omega t + \psi) \tag{625}$$

$$= \frac{kL\hat{I}_0^2 \epsilon^{-2kt}}{2} + \left\{\frac{L(\omega^2 + k^2)^{\frac{1}{2}} \hat{I}_0^2 \epsilon^{-2kt}}{2}\right\} \cos (2\omega t + \psi), \tag{626}$$

since
$$\hat{I} = \hat{I}_0 \epsilon^{-kt}. \tag{627}$$

Thus, the energy changes fall into two parts, one being a simple exponential expression, and the other a periodic double-frequency term with an exponentially decreasing amplitude.

› determine the total energy change it will be necessary to ;rate Eq. (626) between the limits $t = 0$ and $t = \infty$. For purpose it can be put in the form

$$\frac{kL\hat{I}_0^2\epsilon^{-2kt}}{2} + \frac{L(\omega^2 + k^2)^{\frac{1}{2}}\hat{I}_0^2\epsilon^{-2kt}}{2}(\cos 2\omega t \cos\psi - \sin 2\omega t \sin\psi) \quad (628)$$

$$\frac{kL\hat{I}_0^2\epsilon^{-2kt}}{2} + \frac{kL\hat{I}_0^2\epsilon^{-2kt}}{2}\cos 2\omega t + \frac{\omega L\hat{I}_0^2\epsilon^{-2kt}}{2}\sin 2\omega t. \quad (629)$$

:membering that $\displaystyle\int_0^\infty \epsilon^{-2kt}\,dt = \frac{1}{2k}$ (630)

$$\int_0^\infty \epsilon^{-2kt}\cos 2\omega t\,dt = \frac{k}{2(\omega^2 + k^2)} \quad (631)$$

$$\int_0^\infty \epsilon^{-2kt}\sin 2\omega t\,dt = \frac{\omega}{2(\omega^2 + k^2)} \quad (632)$$

$$\int_0^\infty p_L dt = \frac{L\hat{I}_0^2}{2}\left\{\frac{1}{2} + \frac{k^2}{2}\Big/(\omega^2 + k^2) + \frac{\omega^2}{2(\omega^2 + k^2)}\right. \quad (633)$$

$$= \frac{L\hat{I}_0^2}{2}. \quad (634)$$

.is is, of course, in accordance with the physical conditions ı obtain at the instants $t = 0$ and $t = \infty$, for at $t = 0$ there is rent of magnitude

$$i = (\hat{I}_0\epsilon^{-kt}\cos\omega t)_{t=0} \quad (635)$$
$$= \hat{I}_0$$

ıg in the inductance L, and at the instant $t = \infty$, *i.e.*, in ice when the amplitude of the current has become inappre- ɘ, which will, in general, be after a small fraction of a second, $i = 0$. Thus, the total change of current is $\hat{I}_0$ and the change of energy $\frac{L\hat{I}_0^2}{2}$ in accordance with the usual formula.

a precisely similar manner it could be shown that, given an ation of potential across the plates of a condenser of capacity presented by

$$e = \hat{E}_0\epsilon^{-kt}\cos\omega t, \quad (636)$$

otal change of the energy associated with the condenser l be $\frac{1}{2}C\hat{E}_0^2$, since in the course of the oscillation the poten- ılls from $\hat{E}_0$ to 0.

On the other hand, if the oscillation be represented by

$$e = \hat{E}_0\epsilon^{-kt} \sin \omega t, \tag{637}$$

the total change of energy associated with the condenser will be found to be zero by the integration of the energy equation, since the potential of the condenser is zero both at the commencement and at the end of the oscillation. The above integrations will not be given in detail, as they are essentially of the same type as the one already given as an example.

**58. Free Oscillations in an Oscillatory Circuit.**—Suppose that in the circuit shown in Fig. 53 a current is flowing represented by the vector **I**, where

$$\mathbf{I} \cdot \nu = \hat{I}_0\epsilon^{-kt} \cos (\omega t + \psi). \tag{638}$$

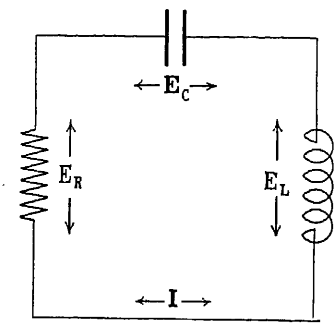


FIG. 53.

For the sum of $\mathbf{E}_R$, $\mathbf{E}_L$, and $\mathbf{E}_C$ the result is

$$-\left\{R\mathbf{I} + L(\omega j - k)\mathbf{I} + \frac{1}{C(\omega j - k)}\mathbf{I}\right\}$$
$$= -\left\{R + L(\omega j - k) + \frac{1}{C(\omega j - k)}\right\}\mathbf{I}. \tag{639}$$

To satisfy Kirchhoff's second law, therefore, an e.m.f. **E** of this magnitude will be required to maintain the assumed current. The case is illustrated in Fig. 54. It should be noted that the angle between $\mathbf{E}_L$ and **I** is $\tan^{-1}\frac{-\omega}{k}$ and that between $\mathbf{E}_C$ and **I** is $\tan^{-1}\frac{\omega}{k}$. Thus, $\mathbf{E}_L$ and $\mathbf{E}_C$ are equally inclined to **I**.

Suppose now that

$$\hat{E}_L = \hat{E}_C, \tag{640}$$

$\hat{E}_L$ and $\hat{E}_C$ being the magnitudes of $\mathbf{E}_L$ and $\mathbf{E}_C$. Then

$$L(\omega^2 + k^2)^{\frac{1}{2}}\hat{I} = \frac{\hat{I}}{C(\omega^2 + k^2)^{\frac{1}{2}}} \tag{641}$$

or

$$LC(\omega^2 + k^2) = 1. \tag{642}$$

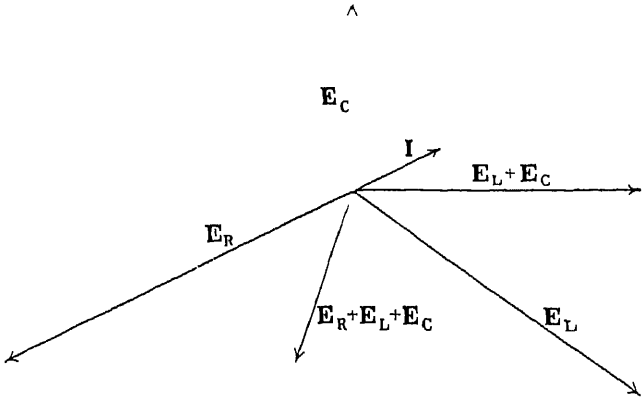


Fig. 54.

If, therefore, $\omega$ has such a value that the above equation is satisfied, *i.e.*, if

$$\omega^2 = \frac{1}{LC} - k^2, \tag{643}$$

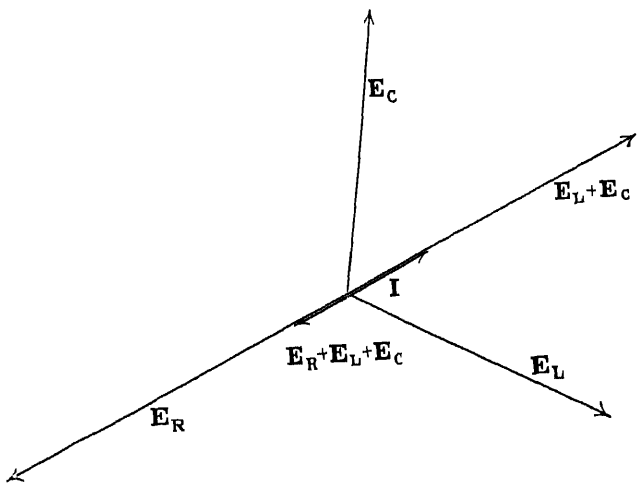


Fig. 55.

$\mathbf{E}_L$ and $\mathbf{E}_C$ will be equal in magnitude. It follows, therefore, since $\mathbf{E}_L$ and $\mathbf{E}_C$ are equally inclined to **I**, that $\mathbf{E}_L + \mathbf{E}_C$ will be in the same direction as **I**, as shown in Fig. 55.

For the value of $\mathbf{E}_L + \mathbf{E}_C$ in terms of $\mathbf{I}$

$$\mathbf{E}_L + \mathbf{E}_C = -\left\{L(\omega j - k) + \frac{1}{C(\omega j - k)}\right\}\mathbf{I} \tag{644}$$

$$= -\left\{L(\omega j - k) + \frac{(\omega j + k)}{C(-\omega^2 - k^2)}\right\}\mathbf{I} \tag{645}$$

$$= -\{L(\omega j - k) - L(\omega j + k)\}, \tag{646}$$

since
$$C(\omega^2 + k^2) = \frac{1}{L}. \tag{647}$$

Thus,
$$\mathbf{E}_L + \mathbf{E}_C = 2kL\mathbf{I}. \tag{648}$$

Under these conditions the e.m.f. required to maintain the assumed current is clearly

$$-(-R + 2kL)\mathbf{I} = \mathbf{E}. \tag{649}$$

Suppose, now, that, in addition to the condition imposed above, namely,
$$LC(\omega^2 + k^2) = 1, \tag{650}$$
the value of k is such that

$$-R + 2kL = 0, \text{ or } k = \frac{R}{2L}. \tag{651}$$

Then $\mathbf{E} = 0$. In other words, *no e.m.f. is required to maintain the current* (see Fig. 56).

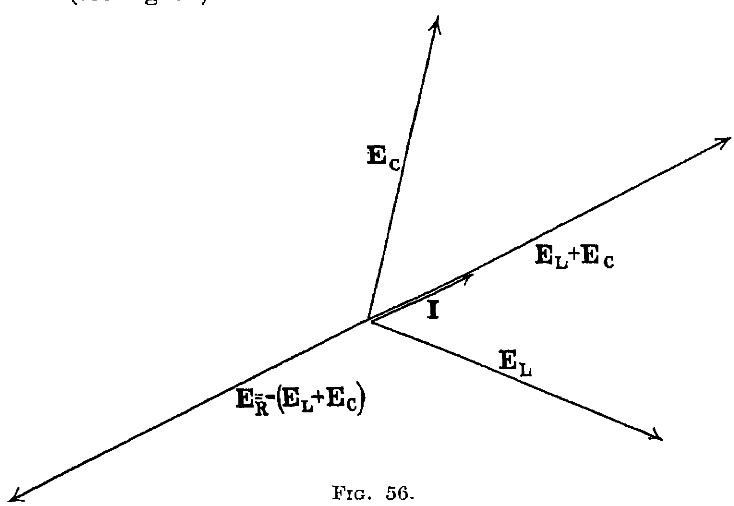


Fig. 56.

Thus, to a current given by

$$i = \mathbf{I} \cdot \mathbf{v} = \hat{I}_0 \epsilon^{-kt} \cos(\omega t + \psi), \tag{652}$$

where
$$k = \frac{R}{2L} \tag{653}$$

and
$$\omega^2 = \frac{1}{LC} - k^2, \tag{654}$$

the circuit illustrated in Fig. 53 offers no resistance, so that no e.m.f. will be required to maintain the current. If, therefore, by any means, which need not yet be specified, a current is caused to flow in the circuit at the instant $t = 0$, then at any subsequent instant $t$ the current will satisfy Kirchhoff's second law, provided

$$i = \hat{I}_0\epsilon^{-kt}\cos(\omega t + \psi), \tag{655}$$

where

$$k = \frac{R}{2L} \tag{656}$$

and

$$\omega^2 = \frac{1}{LC} - k^2. \tag{657}$$

The circuit has, therefore, a natural free period of its own associated with a natural free decrement factor, determined by its electrical constants as shown above. It is for this reason described as an oscillatory circuit, and the current $i$ which results from any electrical disturbance is termed a free oscillation.

The amplitude of the oscillation at the instant $t$ is given by $\hat{I}_0\epsilon^{-kt}$. If

$$T = \frac{2\pi}{\omega} \tag{658}$$

be the period of the oscillation, the amplitude at the instant $t + T$ will be

$$\hat{I}_0\epsilon^{-k(t+T)} = \hat{I}_0\epsilon^{-kt}\epsilon^{-kT}. \tag{659}$$

The ratio of the first of these amplitudes to the second is

$$\frac{1}{\epsilon^{-kT}} = \epsilon^{kT}. \tag{660}$$

The logarithm of this ratio to the base $\epsilon$ is, therefore, $kT$.

*This quantity $kT$, which is a constant, is termed the "logarithmic decrement" of the oscillation.*

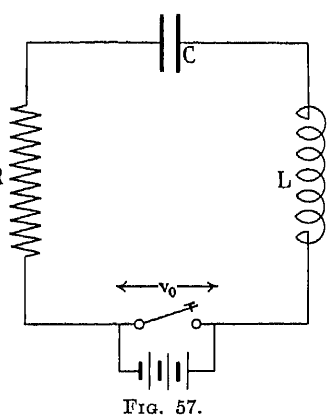


Fig. 57.

**59. The Amplitude and the Phase of a Free Oscillation.**—The constants $\hat{I}_0$ and $\psi$ in the expression

$$I \cdot \nu = \hat{I}_0\epsilon^{-kt}\cos(\omega t + \psi) \tag{661}$$

for the free oscillation of the circuit considered in Par. 58 are as yet undetermined. They will, of course, depend on the nature of the initial disturbance which produces the oscillation. This can be illustrated by considering the case in which the condenser is charged

to a potential $+v_0$ by some suitable source which is short-circuited at the instant $t = 0$ (see Fig. 57). The boundary conditions in this case are

$$i = 0 \text{ when } t = 0 \tag{662}$$
$$e_c = +v_0 \text{ when } t = 0 \tag{663}$$

For $\mathbf{E}_c$ therefore, since

$$i = \hat{I}_0 \epsilon^{-kt} \cos(\omega t + \psi) \tag{664}$$

$$\mathbf{E}_c = -\frac{1}{C(\omega j - k)}\mathbf{I} \tag{665}$$

$$= -\frac{1}{C\sqrt{\omega^2 + k^2}}\epsilon^{j\phi}\mathbf{I} \tag{666}$$

where $$\phi = -\tan^{-1}\frac{\omega}{-k}, \tag{667}$$

*i.e.*, $\phi$ lies between 180 and 270°. Therefore,

$$\mathbf{E}_c = -\frac{\sqrt{LC}}{C}\epsilon^{j\phi}\mathbf{I}, \tag{668}$$

since $$\omega^2 + k^2 = \frac{1}{LC}. \tag{669}$$

i.e. $$e_c = -\frac{\sqrt{LC}}{C}\hat{I}_0\epsilon^{-kt}\cos(\omega t + \psi + \phi) \tag{670}$$

and when $t = 0$

$$(e_c)_{t=0} = -\frac{\sqrt{LC}}{C}\hat{I}_0\cos(\psi + \phi) \tag{671}$$

$$= -\frac{\sqrt{LC}}{C}\hat{I}_0(\cos\psi\cos\phi - \sin\psi\sin\phi) \tag{672}$$

and, since $$\cos\phi = \frac{-k}{\sqrt{\omega^2 + k^2}} = -k\sqrt{LC} \tag{673}$$

and $$\sin\phi = \frac{-\omega}{\sqrt{\omega^2 + k^2}} = -\omega\sqrt{LC} \tag{674}$$

$$(l_c)_{t=0} = kL\hat{I}_0\cos\psi - \omega L\hat{I}_0\sin\psi \tag{675}$$
$$= +v_0. \tag{676}$$

Also, when $t = 0$

$$(i)_{t=0} = \hat{I}_0\cos\psi = 0 \tag{677}$$

$$\therefore\ \psi = \pm\frac{\pi}{2}. \tag{678}$$

From Eq. (675)

$$-\omega L\hat{I}_0\sin\psi = +v_0. \tag{679}$$

Therefore, $$\hat{I}_0 = -\frac{v_0}{\omega L} \tag{680}$$

and $$\psi = \frac{\pi}{2}. \tag{681}$$

Thus, finally $$i = -\frac{V_0}{\omega L}\epsilon^{-kt}\cos\left(\omega t + \frac{\pi}{2}\right) \tag{682}$$

$$= +\frac{V_0}{\omega L}\epsilon^{-kt}\sin\omega t. \tag{683}$$

**60. Forced Damped Oscillations.**—Consider the circuit shown in Fig. 58, which represents an inductance L, a capacity C, and a resistance R in series with a source of exponentially damped

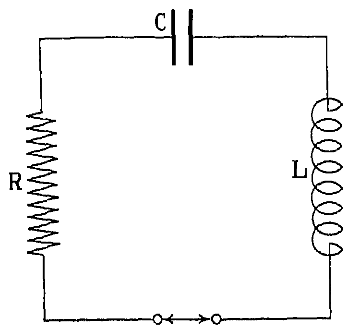


FIG. 58.

e.m.f. For the reasons already given, the current **I** produced by the e.m.f. will be of the same frequency and the same damping factor as **E**, and, applying Kirchhoff's second law to the circuit,

$$\left\{L(\omega j - k) + R + \frac{1}{C(\omega j - k)}\right\}\mathbf{I} = \mathbf{E}, \tag{684}$$

*i.e.,* $$L\left\{(\omega j - k) + \frac{R}{L} + \frac{1}{LC(\omega j - k)}\right\}\mathbf{I} = \mathbf{E}. \tag{685}$$

Now put $$\omega^2 + k^2 = p^2 \tag{686}$$

$$\frac{R}{2L} = k_1 \tag{687}$$

$$\frac{1}{LC} = p_1^2 \tag{688}$$

and $$\omega_1^2 = p_1^2 - k_1^2, \tag{689}$$

so that $\frac{\omega_1}{2\pi}$ is the natural frequency of the circuit and $k_1$ is its natural damping factor. Then

$$L\left\{(\omega j - k) + 2k_1 - \frac{p_1^2}{p^2}(\omega j + k)\right\}\mathbf{I} = \mathbf{E} \tag{690}$$

*i.e.,* $$L\left\{\omega j\left(1 - \frac{p_1^2}{p^2}\right) + 2k_1 - k\left(1 + \frac{p_1^2}{p^2}\right)\right\}\mathbf{I} = \mathbf{E}. \tag{691}$$

The condition for the vanishing of the j term is clearly $p_1^2 = p^2$. Thus $1 - \frac{p_1^2}{p^2}$ can be regarded as a measure of the degree of distuning of the circuit. Writing d for this,

$$L\{\omega j d + 2(k_1 - k) + dk\}\mathbf{I} = \mathbf{E}, \tag{692}$$

a form which serves to emphasize the effect of tuning the circuit, *i.e.*, of decreasing the value of d to zero.

For abbreviation put

$$\omega d = b \tag{693}$$

$$2(k_1 - k) + dk = a \tag{694}$$

and

$$r^2 = a^2 + b^2. \tag{695}$$

Then

$$\mathbf{I} = \frac{\mathbf{E}}{L(a + jb)} \tag{696}$$

and

$$i = \frac{\hat{E}_0\epsilon^{-kt}}{Lr^2}(a \cos \omega t + b \sin \omega t). \tag{697}$$

For $e_C$

$$\mathbf{E}_C = -\frac{1}{C(\omega j - k)}\mathbf{I} \tag{698}$$

$$= \frac{\omega j + k}{Cp^2}\mathbf{I} \tag{969}$$

$$= \frac{1}{LCp^2r^2}(a - jb)(k + \omega j)\mathbf{E} \tag{700}$$

$$\therefore\ e_C = \frac{\hat{E}_0\epsilon^{-kt}}{LCp^2r^2}\{(ak + \omega b)\cos \omega t + (bk - a\omega)\sin \omega t\}\cdot \tag{701}$$

The above, however, cannot be the complete solution, for, if it were, the initial conditions would be

$$(i)_{t=0} = \frac{a}{r^2}\frac{\hat{E}_0}{L} \tag{702}$$

$$(e_C)_{t=0} = \frac{ak + \omega b}{r^2}\frac{\hat{E}_0}{LCp^2}, \tag{703}$$

whereas the physical conditions of the circuit require that both i and $e_C$ shall be zero when $t = 0$. To satisfy these conditions, therefore, it must be assumed that, in addition, the circuit is thrown into a state of free oscillation, the initial conditions of which just balance the initial conditions of the forced oscillation, *i.e.*, the initial conditions of the free oscillation are given by

$$(i)_{t=0} = -\frac{a}{r^2}\frac{\hat{E}_0}{L} \tag{704}$$

$$(e_C)_{t=0} = \frac{-(ak + \omega b)}{r^2}\frac{\hat{E}_0}{LCp^2}. \tag{705}$$

It will be convenient to express the free oscillation in the form

$$\mathbf{I} = \frac{1}{L(a_1 + jb_1)}\mathbf{E}_1, \tag{706}$$

where

$$\mathbf{E}_1 \cdot \mathbf{v} = \hat{E}_0\epsilon^{-k_1t}\cos \omega_1 t. \tag{707}$$

In this expression $a_1$ and $b_1$ are the two unknown constants to be determined. For i,

$$i = \mathbf{I}\cdot\mathbf{v} = \frac{\hat{E}_0}{L}\epsilon^{-k_1t}\left(\frac{a_1}{r_1^2}\cos \omega_1 t + \frac{b_1}{r_1^2}\sin \omega_1 t\right)\cdot \tag{708}$$

Comparing Eq. (706) with Eq. (696) it is seen that $a_1$ and $b_1$ must satisfy

$$\frac{a_1}{r_1^2} = -\frac{a}{r^2} \qquad (709)$$

$$\frac{a_1k_1 + \omega_1b_1}{p_1^2r_1^2} = -\frac{ak + \omega b}{p^2r^2}, \qquad (710)$$

*i.e.,*

$$\frac{a_1k_1}{r_1^2} + \frac{\omega_1b_1}{r_1^2} = -\frac{p_1^2}{p^2}\frac{ak + \omega b}{r^2} \qquad (711)$$

$$= -\frac{(1 - d)(ak + \omega b)}{r^2} \qquad (712)$$

$$\therefore \quad \frac{b_1}{r_1^2} = \frac{1}{\omega_1{}^2r^2}[a\{(k_1 - k) + dk\} - b(\omega - \omega d)] \qquad (713)$$

and, since $\quad b = \omega d \qquad (714)$

and $\quad a = 2(k_1 - k) + dk, \qquad (715)$

$$\frac{b_1}{r_1^2} = \frac{1}{\omega_1r^2}\left[\frac{a}{2}(a + dk) - b(\omega - b)\right]. \qquad (716)$$

For the free oscillation, therefore, since

$$i = \frac{\hat{E}_0}{L}\epsilon^{-k_1t}\left\{\frac{a_1}{r_1^2}\cos\omega_1t + \frac{b_1}{r_1^2}\sin\omega_1t\right\} \qquad (717)$$

$$i = \frac{\hat{E}_0}{Lr^2}\epsilon^{-k_1t}\left[-a\cos\omega_1t + \frac{1}{\omega_1}\left\{\frac{a(a + dk)}{2} + b(b - \omega)\right\}\sin\omega_1t\right]. \qquad (718)$$

For the forced oscillation Eq. (697)

$$i = \frac{\hat{E}_0}{Lr^2}\epsilon^{-kt}\,(a\cos\omega t + b\sin\omega t). \qquad (719)$$

The total solution, which is the sum of the free and the forced oscillations is, therefore, given by

$$i = \frac{a\epsilon^{-kt}\hat{E}_0}{Lr^2}\cos\omega t + \frac{b\epsilon^{-kt}\hat{E}_0}{Lr^2}\sin\omega t$$
$$- \frac{a\epsilon^{-k_1t}\hat{E}_0}{Lr^2}\cos\omega_1t + \left\{\frac{a(a + dk)}{2} + b(b - \omega)\right\}\frac{\epsilon^{-k_1t}}{\omega_1Lr^2}\hat{E}_0\sin\omega_1t \qquad (720)$$

where, repeating the abbreviations for convenience,

$$p_1^2 = \frac{1}{LC} \qquad (721)$$

$$k_1 = \frac{R}{2L} \qquad (722)$$

$$\omega_1^2 = p_1^2 - k_1^2 \qquad (723)$$

$$p^2 = \omega^2 + k^2 \qquad (724)$$

$$a = 2(k_1 - k) + dk \qquad (725)$$

$$b = \omega d \qquad (726)$$

$$r^2 = a^2 + b^2 \qquad (727)$$

$$d = 1 - \frac{p_1^2}{p^2}. \qquad (728)$$

The most interesting case is that in which the circuit is tuned to resonance, *i.e.,*

$$p_1^2 = p^2 \text{ or } d = 0, \qquad (729)$$

which will give $\quad a = 2(k_1 - k)\ b = 0$ and $r^2 = a^2, \qquad (730)$

so that $$i = \frac{\epsilon^{-kt}\hat{E}_0 \cos \omega t}{La} - \frac{\epsilon^{-k_1t}\hat{E}_0 \cos \omega_1 t}{La} + \frac{\epsilon^{-k_1t}}{2\omega_1 L}\hat{E}_0 \sin \omega_1 t. \qquad (731)$$

If, in addition, the resistance R is varied so as to make

$$\frac{R}{2L} = k_1 = k, \qquad (732)$$

then $$a = 0 \qquad (733)$$

and $$\omega = \omega_1. \qquad (734)$$

The first two terms become infinite in synchronism but in phase opposition and the remainder is

$$i = \frac{\epsilon^{-kt}\hat{E}_0}{2\omega L} \sin \omega t. \qquad (735)$$

**61. The Production of Undamped or Continuous Oscillations.** The amount of space devoted in the preceding paragraphs to the

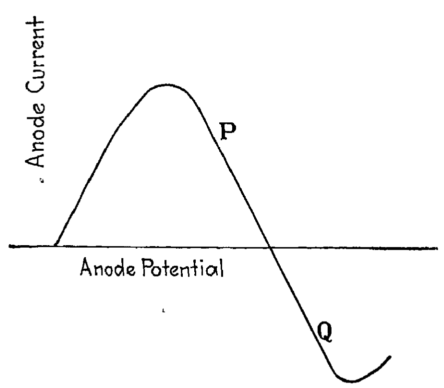


FIG. 59.

analysis of damped oscillations may seem excessive in view of the decreasing importance of such oscillations as compared with continuous or undamped oscillations, particularly in relation to the development of radio telegraphy. A knowledge of the theory of damped oscillations is still very desirable, however, since some of the most widely used methods of generating continuous oscillations consist essentially of the "undamping" of a damped oscillation, that is, of the inserting into an oscillatory circuit of something which annuls the effect of its resistance and which may, therefore, be suitably described as a negative resistance.

One method of obtaining this result is to connect in parallel with the oscillating circuit a conductor whose current-potential

characteristic has a negative slope. The dynatron, for instance,[1] has a characteristic of this nature. Under suitable conditions of anode and grid potential the relation between the anode current and the potential difference acting in the anode circuit can be represented by a curve of the form shown in Fig. 59. The part PQ can be considered as approximately a straight line having the equation

$$i_0 = A - \frac{v_0}{B}, \tag{736}$$

A and B being constants.

If, now, an alternating potential difference of instantaneous value v be superimposed on $v_0$, and if i be the instantaneous value of the alternating current which results, then

$$i + i_0 = A - \frac{v_0}{B} - \frac{v}{B}, \tag{737}$$

*i.e.,*

$$i = -\frac{v}{B}. \tag{738}$$

Thus, to an alternating potential superimposed on the steady or mean potential $v_0$, the dynatron behaves as if it were a negative resistance of magnitude B.

Consider now the effect of connecting such a negative resistance in parallel with the condenser of an oscillatory circuit, as shown in

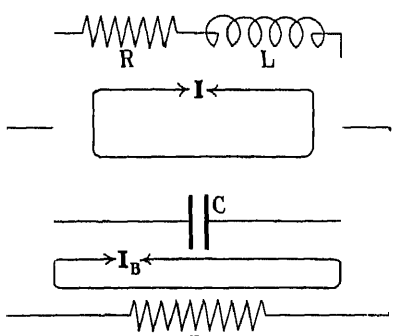


FIG. 60.

Fig. 60. If I and $I_B$ be vectors representing the current flowing in the oscillatory circuit and the circuit $-B$, C, $-B$ respectively, then, assuming the free oscillation to be of frequency $\omega$ and damping factor k,

[1] See *Proc. Inst. of Radio Engineers,* vol. 16, which contains an article by A. W. Hull on this subject.

$$\left\{(\omega j - k)L + R + \frac{1}{(\omega j - k)C}\right\}I - \frac{I_B}{(\omega j - k)C} = 0 \quad (739)$$

$$-BI_B + \{(\omega j - k)L + R\}I = 0. \quad (740)$$

From the second of these

$$I_B = \frac{(\omega j - k)L + R}{B} I. \quad (741)$$

Substituting this result in Eq. (739),

$$\left[(\omega j - k)L + R - \frac{1}{(\omega j - k)C}\left\{\frac{(\omega j - k)L + R}{B}\right\}\right]I = 0, \quad (742)$$

*i.e.*,

$$\left\{(\omega j - k)L + \left(R - \frac{L}{BC}\right) + \frac{1}{(\omega j - k)C}\left(1 + \frac{R}{B - R}\right)\right\}I = 0. \quad (743)$$

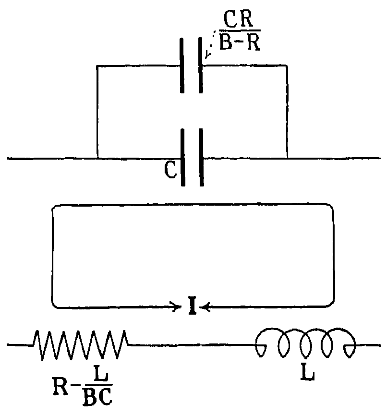


FIG. 61.

Equation (743) shows that the circuit of Fig. 60 can virtually be replaced by the circuit shown in Fig. 61. As in Par. 58, the free oscillation of such a circuit would have a damping coefficient

$$k = \frac{R - \dfrac{L}{BC}}{2L} \quad (744)$$

and a frequency $\frac{\omega}{2\pi}$ where

$$\omega^2 = \frac{1}{LC\left(1 + \dfrac{R}{B - R}\right)} - k^2. \quad (745)$$

If, now, B be given a value such that

$$R - \frac{L}{BC} = 0, \tag{746}$$

*i.e.*,

$$B = \frac{L}{CR}, \tag{747}$$

then

$$k = 0 \tag{748}$$

and

$$\omega^2 = \frac{1}{LC\left(1 + \frac{R}{B - R}\right)}. \tag{749}$$

Under these conditions, therefore, the natural damping of the circuit has been annulled and continuous oscillations of the frequency given by Eq. (749) will result.

It is of interest to trace the source of the energy required for the maintenance of the continuous oscillations.

If $i_0$ and $v_0$ be the current through B and the potential across it before oscillations occur, then

$$i_0 = A - \frac{v_0}{B} \tag{750}$$

or

$$v_0 = B(A - i_0). \tag{751}$$

The power being consumed in B is, therefore,

$$v_0 i_0 = BAi_0 - i_0^2B = P. \tag{752}$$

In the oscillating condition the alternating current represented by $\mathbf{I}_B$ will be superimposed on i. Putting $i_B = \mathbf{I}_B \cdot \mathbf{v}$, then

$$v + v_0 = B\{A - (i_0 + i_B)\} \tag{753}$$

and for the instantaneous power consumed in B,

$$(i_0 + i)(v_0 + v) = BA(i_0 + i_B) - (i_0 + i_B)^2B. \tag{754}$$

The mean power consumed will, therefore, be the mean value of the expression on the right-hand side of Eq. (754) *i.e.*, the mean value of

$$BAi_0 + BAi_B - i_0^2B - 2i_0i_BB - i_B^2B. \tag{755}$$

Since $i_B$ is an alternating current, its mean value over a period is zero, so that the mean values of the second and fourth terms in the above expression are zero. As shown in Par. 41, the mean value of $i_B^2B$ is $\frac{\hat{I}_B^2B}{2}$, $\hat{I}_B$ being the magnitude of $\mathbf{I}_B$.

Thus, the mean power consumed when oscillations occur is

$$BAi_0 - i_0^2B - \frac{\hat{I}_B^2B}{2} = P - \frac{\hat{I}_B^2B}{2}, \tag{756}$$

which is less than the power consumed in the stationary condition by an amount $\frac{\hat{I}_B^2B}{2}$.

Now from Eq. (741)

$$I_B = \frac{\omega jL + R}{B} I, \tag{757}$$

so that
$$\hat{I}_B{}^2 = \frac{\omega^2L^2 + R^2}{B^2} \hat{I}^2 \tag{758}$$

and
$$\hat{I}_B{}^2B = \frac{\omega^2L^2 + R^2}{B} \hat{I}^2. \tag{759}$$

Also,
$$\omega^2 = \frac{1}{LC\left(1 + \frac{R}{B - R}\right)} \tag{760}$$

$$= \frac{B - R}{LCB} \tag{761}$$

$$= \frac{\frac{L}{CR} - R}{LC \cdot \frac{L}{CR}} \tag{762}$$

$$= \frac{L - CR^2}{L^2C}. \tag{763}$$

Therefore,
$$(\omega^2L^2 + R^2) = \left(\frac{L - CR^2}{C} + R^2\right)\frac{1}{B} \tag{764}$$

$$= \frac{L}{C} \cdot \frac{CR}{L} \tag{765}$$

$$= R. \tag{766}$$

Thus, finally,
$$\frac{\hat{I}_B{}^2B}{2} = \frac{\hat{I}^2R}{2}. \tag{767}$$

In other words, the source of potential which maintains the conductor B in a negative-resistance condition supplies to that conductor a certain amount of power. When oscillations occur in the oscillatory circuit, the power consumed by the conductor B is decreased by an amount exactly equal to the power consumed by the resistance of the oscillatory circuit. That is to say, power is diverted to the oscillatory circuit from the conductor B, which thus appears to function as a sort of agent between the source of potential and the oscillatory circuit.

**62. The Generation of Continuous Oscillations by Means of the Thermionic Valve.**—The case of the valve generator is essentially similar to the above, though the mechanism is somewhat different. The process will be illustrated by reference to the typical generating circuit illustrated in Fig. 62.

It is well known that changes of anode potential, grid potential, and anode current (represented by $v_a$, $v_g$, and $i_a$ respectively) are related by an equation of the form

$$i_a = av_a + gv_g, \tag{768}$$

a and g being constants for a given valve, it being understood that these changes are all within the region over which the

characteristics of the valve can be regarded as straight lines, or approximately so.

Putting
$$\frac{g}{a} = \mu \tag{769}$$

and
$$a = \frac{1}{R_a}, \tag{770}$$

then
$$i_a = \frac{v_a + \mu v_g}{R_a}. \tag{771}$$

The meaning of the above equation must be clearly understood. It is that, irrespective of the initial values of anode potential,

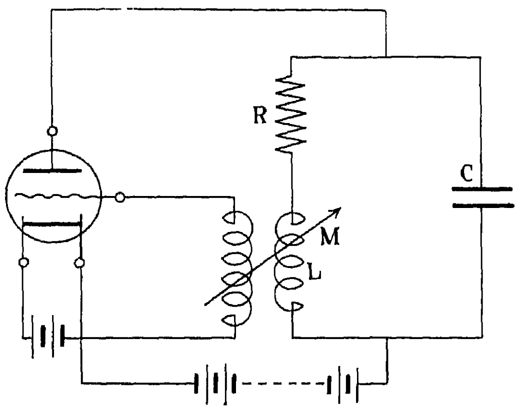


FIG. 62.

grid potential, and anode current, a change of anode potential of amount $v_a$, together with a change of grid potential of amount $v_g$, will produce a change of anode current of the amount given by the equation. For instance, if $v_a$ and $v_g$ be the instantaneous values of alternating potentials operating in the anode and the grid circuits respectively, then $i_a$ will be the instantaneous value of the alternating component of the anode current. If these alternating quantities be represented by vectors in the usual way, the result is the vector equation

$$\mathbf{I}_a = \frac{\mathbf{V}_a + \mu \mathbf{V}_g}{R_a}. \tag{772}$$

Considering this last equation, it is clear that the anode circuit of the valve is electrically equivalent to the simpler circuit illustrated in Fig. 63, and that to any circuit connected between the negative end of the filament and the anode the valve will behave as a source of potential of magnitude $\mu v_g$ with an internal resistance $R_a$. Thus, the generating circuit illustrated in Fig. 62 is electrically equivalent to the arrangement shown in Fig. 64.

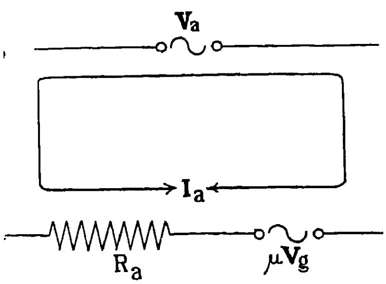


FIG. 63.

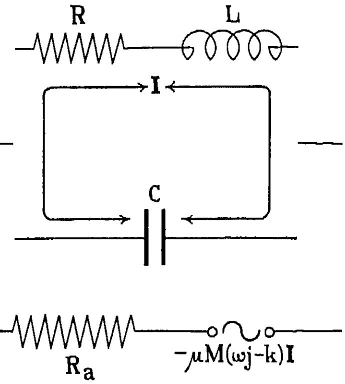


FIG. 64.

If **I** and $\mathbf{I}_a$ be vectors representing respectively the free oscillation in the oscillatory circuit and the alternating component of the anode current, these vectors being of angular velocity $\omega$ and damping coefficient k, then for the circuits C, R, L, C and $R_a$, R, L, $R_a$

$$\left\{(\omega j - k)L + R + \frac{1}{(\omega j - k)C}\right\}\mathbf{I} - \frac{\mathbf{I}_a}{(\omega j - k)C} = 0 \qquad (773)$$

$$R_a\mathbf{I}_a + \{R + L(\omega j - k)\}\mathbf{I} = \mu V_g \qquad (774)$$

$$= -\,\mu M(\omega j - k)\mathbf{I}, \qquad (775)$$

so that

$$-\frac{\mathbf{I}_a}{(\omega j - k)C} = \left\{\frac{R}{CR_a(\omega j - k)} + \frac{1}{CR_a} - \frac{\mu M}{CR_a}\right\}\mathbf{I}. \qquad (776)$$

Substituting this in Eq. (773),

$$\left\{(\omega j - k)L + \left(R + \frac{L}{CR_a} - \frac{\mu M}{CR_a}\right) + \frac{\left(1 + \frac{R}{R_a}\right)}{(\omega j - k)C}\right\}\mathbf{I} = 0, \qquad (777)$$

*i.e.*,

$$\left\{(\omega j - k)L + \left(\frac{R + L}{CR_a} - \frac{\mu M}{CR_a}\right) + \frac{1}{(\omega j - k)C(1 - \frac{R_a}{R + R_a}}\right\}\mathbf{I} = 0. \qquad (778)$$

The oscillatory circuit is, therefore, effectively shown in Fig. 65. For such a circuit the natural damping coefficient is

$$k = \frac{\left(R + \frac{L}{CR_a} - \frac{\mu M}{CR_a}\right)}{2L} \tag{779}$$

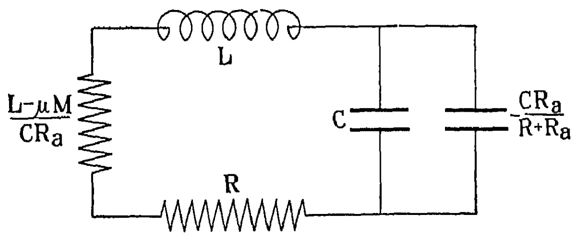


Fig. 65.

and the natural frequency $\frac{\omega}{2\pi}$, where

$$\omega^2 = \frac{1}{LC\left(1 - \frac{R}{R + R_a}\right)}. \tag{780}$$

If, now, M be given the value $\frac{(CRR_a + L)}{\mu}$ then

$$CRR_a + L - \mu M = 0, \tag{781}$$

*i.e.*, k = 0, and undamped or continuous oscillations will be produced, their frequency being given by $\frac{\omega}{2\pi}$, where

$$\omega^2 = \frac{1}{LC\left(1 - \frac{R}{R + R_a}\right)}. \tag{782}$$

As in the case considered in Par. 61, it can easily be shown that, when in the oscillating condition, the power consumed by the valve is diminished by an amount exactly equal to that consumed by the resistance of the oscillating circuit. The valve is thus an agent which receives power from its own anode battery and passes on a certain amount of it to the oscillatory circuit. It can further be shown that, on the assumption that the total anode current, *i.e.*, the fixed mean value plus the alternating component, does not fall below zero nor exceed the saturation value for the given valve, the maximum power handed on by the valve to the oscillatory circuit is 50 per cent of that which it receives from the anode battery.

**63. Free Oscillations in Coupled Circuits.**—1. Consider the magnetically coupled oscillatory circuits illustrated in Fig. 65, and assume that the currents $i_1$ and $i_2$ shown in the figure are the instantaneous values of the free oscillations proper to the circuits due to their own electrical constants and to the mutual induction M between them. Remembering that the true definition of a free oscillation is a current that requires no external source of e.m.f. to maintain it, then, since $i_1$ and $i_2$ are free oscillations, the

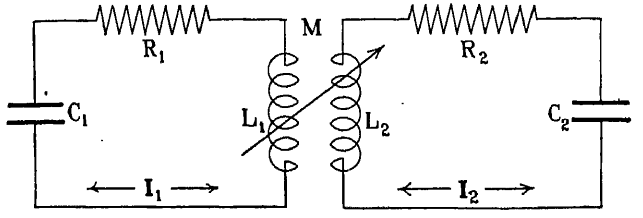


FIG. 66.

sum of the back e.m.fs. due to them must be zero at every instant in the two closed circuits. Thus, for the first circuit

$$\mathbf{E}_{L1} + \mathbf{E}_{R1} + \mathbf{E}_{C1} + \mathbf{E}_{M1} = 0, \tag{783}$$

where $\mathbf{E}_{M1}$ is the vector which represents the instantaneous value of $M\frac{di_2}{dt}$, *i.e.*, the back e.m.f. induced in this circuit by the current $i_2$ flowing in the second circuit.

Now it has been shown (Par. 31) that, if a vector equation of this type is to be true at every instant, the component vectors must all be of the same type as regards angular velocity and decrement factor. Thus, the vector $\mathbf{E}_{M1}$, which represents $M\frac{di_2}{dt}$, must have the same angular velocity and the same decrement factor as the vector $\mathbf{E}_{L1}$, which represents $L\frac{di_1}{dt}$. It follows that the currents $i_1$ and $i_2$ must be of the same frequency and the same decrement. It can, therefore, be said that, if free oscillations exist at all, they will be of the same frequency and the same decrement in each circuit. Taken together, therefore, the two circuits can be said to constitute a single electrical system.

If the currents $i_1$ and $i_2$ be represented by the vectors $I_1$ and $I_2$, where

$$I_1 \cdot \nu = \hat{I}_1\epsilon^{-kt} \cos (\omega t + \psi_1) \tag{784}$$

and

$$I_2 \cdot \nu = \hat{I}_2\epsilon^{-kt} \cos (\omega t + \psi_2), \tag{785}$$

then for both circuits

$$\mathbf{E}_L + \mathbf{E}_R + \mathbf{E}_C + \mathbf{E}_M = 0 \tag{786}$$

and, substituting for these vectors their expressions in terms of $\mathbf{I}_1$ and $\mathbf{I}_2$,

$$\left\{L_1(\omega j - k) + R_1 + \frac{1}{C_1(\omega j - k)}\right\}\mathbf{I}_1 + M(\omega j - k)\mathbf{I}_2 = 0 \tag{787}$$

for the first circuit, and

$$\left\{L_2\,(\omega j - k) + R_2 + \frac{1}{C_2(\omega j - k)}\right\}\mathbf{I}_2 + M(\omega j - k)\mathbf{I}_1 = 0 \tag{788}$$

for the second.

From the first of these

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = -\frac{M(\omega j - k)}{L_1(\omega j - k) + R_1 + \dfrac{1}{C_1(\omega j - k)}} \tag{789}$$

and from the second

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = -\frac{L_2(\omega j - k) + R_2 + \dfrac{1}{C_2(\omega j - k)}}{M(\omega j - k)} \tag{790}$$

Therefore,

$$\left\{L_1(\omega j - k) + R_1 + \frac{1}{C_1(\omega j - k)}\right\}\left\{L_2(\omega j - k) + R_2 + \frac{1}{C_2(\omega j - k)}\right\} - M^2(\omega j - k)^2 = 0. \tag{791}$$

Two further deductions can now be made. First, from Eq. (791) it appears that $\omega$ and k are functions only of the electrical constants of the two circuits and of the mutual induction between them, and are thus independent of the amplitudes and the phases of the oscillations. Secondly, assuming that $\omega$ and k can be obtained as known functions of the constants of the circuits, then Eq. (789) is of the form

$$\mathbf{I}_2 = (a + jb)\mathbf{I}_1,$$

where a and b are constants for the given circuits. Thus, while the magnitudes and phases of the currents $\mathbf{I}_1$ and $\mathbf{I}_2$ will depend on the initial conditions which produce the oscillation, the phase difference between $\mathbf{I}_1$ and $\mathbf{I}_2$ and the ratios of their amplitudes are constants for the given circuits.

Dividing Eq. (791) throughout by $L_1L_2$ and putting

$$\frac{R_1}{2L_1} = k_1 \tag{792}$$

$$\frac{R_2}{2L_2} = k_2 \tag{793}$$

$$\frac{1}{L_1C_1} = p_1^2 \quad \text{and} \quad \omega_1^2 = p_1^2 - k_1^2 \tag{794}$$

$$\frac{1}{L_2C_2} = p_2^2 \quad \text{and} \quad \omega_2^2 = p_2^2 - k_2^2 \tag{795}$$

$$\frac{M^2}{L_1L_2} = \mu^2, \tag{796}$$

so that $k_1$ and $k_2$ are natural damping factors of the two circuits considered separately, and $\omega_1$ and $\omega_2$ are the natural free periods of the two circuits, the result is

$$\{(\omega j - k)^2 + 2k_1(\omega j - k) + (\omega_1^2 + k_1^2)\}\{(\omega j - k)^2 + 2k_2(\omega j - k) + (\omega_2^2 + k_2^2)\} - \mu^2(\omega j - k)^4 = 0. \tag{797}$$

This last, being a fourth-power equation in $(\omega j - k)$, can be expressed as the product of two quadratic factors, which may be written in the form

$$\{(\omega j - k)^2 + 2k'(\omega j - k) + (\omega'^2 + k'^2)\}\{(\omega j - k)^2 + 2k''(\omega j - k) + (\omega''^2 + k''^2)\} = 0. \tag{798}$$

This equation will be satisfied if either of the two factors is zero. Equating the first to zero,

$$(\omega j - k)^2 + 2k'(\omega j - k) + (\omega'^2 + k'^2) = 0, \tag{799}$$

*i.e.,*

$$-\omega^2 + k^2 - 2kk' + (\omega'^2 + k'^2) = 0 \tag{800}$$

and

$$-2k\omega + 2k'\omega = 0. \tag{801}$$

Therefore,

$$k = k' \tag{802}$$

$$\omega = \omega'. \tag{803}$$

In a similar manner, by equating the second factor to zero,

$$k = k'' \tag{804}$$

$$\omega = \omega''. \tag{805}$$

It appears, therefore, that any current

$$i_1 = \hat{I}_1\epsilon^{-kt} \cos (\omega t + \psi_1) \tag{806}$$

flowing in the first circuit will satisfy the necessary condition for a free oscillation, provided $k = k'$ and $\omega = \omega'$, or provided $k = k''$ and $\omega = \omega''$, where $k'$ and $\omega'$ and $k''$ and $\omega''$ are the roots of the fourth-power Eq. (797). Moreover, since the sum of the back e.m.fs. due to either of these currents in the closed circuit

is zero, the sum of the back e.m.fs. due to both the currents flowing simultaneously will also be zero. The most general solution for $i_1$ is, therefore, the sum of two currents of frequencies $\omega'$ and $\omega''$ with corresponding damping factors $k'$ and $k''$, *i.e.*,

$$i_1 = \hat{I}_1'\epsilon^{-k't}\cos(\omega't + \psi_1') + \hat{I}_1''\epsilon^{-k''t}\cos(\omega''t + \psi_1'') \quad (807)$$

$$= \mathbf{I}_1' \cdot \boldsymbol{\nu} + \mathbf{I}_1'' \cdot \boldsymbol{\nu}. \quad (808)$$

Further considering Eq. (789),

$$a' + jb' = -\frac{L_1}{M}\frac{(\omega'j - k')^2 + 2k_1(\omega'j - k') + (\omega_1^2 + k_1^2)}{(\omega'j - k')} \quad (809)$$

is the relation between $\mathbf{I}_1'$ and $\mathbf{I}_2'$ and, similarly,

$$a'' + jb'' = -\frac{L_1}{M}\frac{(\omega''j - k'')^2 + 2k_1(\omega''j - k'') + (\omega_1^2 + k_1^2)}{(\omega''j - k'')} \quad (810)$$

is the relation between $\mathbf{I}_1''$ and $\mathbf{I}_2''$, so that the currents in the second circuit are given by

$$i_2 = (a' + jb')\mathbf{I}_1' \cdot \boldsymbol{\nu} + (a'' + jb'')\mathbf{I}'' \cdot \boldsymbol{\nu} \quad (811)$$

or, in scalar form,

$$i_2 = r'\hat{I}_1'\epsilon^{-k't}\cos(\omega't + \psi_1' + \theta') + r''\hat{I}_1''\epsilon^{-k''t}\cos(\omega''t + \psi_1'' + \theta''), \quad (812)$$

where $$a' + jb' = r'\epsilon^{j\theta'} \quad (813)$$

and $$a'' + jb'' = r''\epsilon^{j\theta''}. \quad (814)$$

For the complete expression of the general solution in terms of the electrical constants of the circuits it would be necessary to solve the fourth-power Eq. (797). The solution of this equation is, however, a matter of considerable difficulty, and is beyond the scope of this book. For the full discussion of the general case the reader is referred to "Electric Oscillations and Electric Waves," by G. W. Pierce. There are, however, certain special cases which yield a comparatively simple solution.

2. *Special Cases*.—(*a*) Circuits of Negligible Damping.—If $k_1 = k_2 = 0$, Eq. (797) becomes

$$(-\omega^2 + \omega_1^2)(-\omega^2 + \omega_2^2) - \mu^2\omega^4 = 0, \quad (815)$$

*i.e.*, $$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \mu^2\omega^4 = 0 \quad (816)$$

or $$\omega^4 - \frac{\omega_1^2 + \omega_2^2}{1 - \mu^2}\omega^2 + \frac{\omega_1^2\omega_2^2}{1 - \mu^2} = 0. \quad (817)$$

Therefore,

$$\omega^2 = \frac{(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(1 - \mu^2)\omega_1^2\omega_2^2}}{2(1 - \mu^2)}, \quad (818)$$

*i.e.*,

$$\omega'^2 = \frac{(\omega_1^2 + \omega_2^2) + \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(1 - \mu^2)(\omega_1^2\omega_2^2)}}{2(1 - \mu^2)} \qquad (819)$$

and

$$\omega''^2 = \frac{(\omega_1^2 + \omega_2^2) - \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(1 - \mu^2)\omega_1^2\omega_2^2}}{2(1 - \mu^2)}. \qquad (820)$$

Also, from Eq. (809)

$$(a' + jb') = \frac{L_1}{M}\,\frac{-\omega'^2 + \omega_1^2}{-\omega'^2} \qquad (821)$$

$$= -\frac{L_1}{M}\,\frac{\omega'^2 - \omega_1^2}{\omega'^2}. \qquad (822)$$

Therefore,
$$b' = 0 \qquad (823)$$

and
$$a' = -\frac{L_1}{M}\,\frac{\omega'^2 - \omega_1^2}{\omega'^2}. \qquad (824)$$

Similarly, from Eq. (809)

$$b'' = 0 \qquad (825)$$

$$a'' = -\frac{L_1}{M}\,\frac{\omega''^2 - \omega_1^2}{\omega''^2}. \qquad (826)$$

Or, alternatively from Eq. (790),

$$b'' = 0 \qquad (827)$$

$$a'' = -\frac{M}{L_2}\,\frac{\omega''^2}{\omega''^2 - \omega_2^2}. \qquad (828)$$

It appears, therefore, that, when the damping of the circuits is negligible, each oscillation in the first circuit is either in phase or in antiphase with the corresponding oscillation in the second circuit. To discriminate between these possibilities, the relative magnitudes of the four quantities $\omega'$, $\omega_1$, $\omega''$, $\omega_2$ must be considered, remembering that the limits of $\mu$ are 0 and 1.

For $\omega'$ and $\omega_1$ there is no difficulty, since Eq. (819) shows that the lower limit of $\omega'$ is $\omega_1$ when $\mu = 0$, and that the upper limit is $\infty$ when $\mu = 1$. Therefore $\omega'$ is either equal to or greater than $\omega_1$, so that a′ is either zero or negative. The two oscillations of frequency $\frac{\omega'}{2\pi}$ are thus always in antiphase with each other.

Considering the oscillations of frequency $\frac{\omega''}{2\pi}$, however, it is found that, while the limit of $\omega''$ when $\mu = 0$ is $\omega_2$, the other limit corresponding to $\mu = 1$ appears to be indeterminate. The limit must, therefore, be found by the usual method of differentiating

the top and the bottom of the expression for $\omega''$, *i.e.*, Eq. (820) and then inserting the limiting value of $\mu$, *i.e.*,

$$\frac{\text{lt}\cdot\omega''}{\mu = 1} = \frac{\text{lt}}{\mu = 1}\cdot\frac{\omega_1{}^2\omega_2{}^2}{\sqrt{(\omega_1{}^2 + \omega_2{}^2)^2 - 4(1 - \mu^2)\omega_1{}^2\omega_2{}^2}} \tag{829}$$

$$= \omega_2{}^2\cdot\frac{\omega_1{}^2}{\omega_1{}^2 + \omega_2{}^2}. \tag{830}$$

Thus the limiting value of $\omega''$ is less than $\omega_2\cdot$ Therefore $\omega''$ is always either equal to or less than $\omega_2$. $\cdot$ It follows from Eq. (828) that a$''$ is either infinite or positive. The oscillations of this frequency are, therefore, always in phase.

(*b*) Isochronous Circuits of Negligible Damping.—If in, addition to the condition

$$k_1 \approx k_2 \approx 0, \tag{831}$$

it is true that
$$p_1 = p_2 = p_0, \tag{832}$$

which also implies
$$\omega_1 = \omega_2 = \omega_0, \tag{833}$$

then from Eq. (818)
$$\omega^2 = \frac{\omega_0{}^2 \pm \mu\omega_0{}^2}{1 - \mu^2} \tag{834}$$

$$\therefore\quad \omega'^2 = \frac{1}{1 - \mu}\omega_0{}^2 \tag{835}$$

and
$$\omega''^2 = \frac{1}{1 + \mu}\omega_0{}^2. \tag{836}$$

Further, a and b have the values

$$a' = -\frac{L_1}{M}\cdot\mu = -\sqrt{\frac{L_1}{L_2}},\quad b' = 0 \tag{837}$$

$$a'' = -\frac{L_1}{M}\cdot -\mu = \sqrt{\frac{L_1}{L_2}},\quad b'' = 0. \tag{838}$$

As in the previous case, therefore, the oscillations of frequency $\frac{\omega'}{2\pi}$ are always in antiphase and those of frequency $\frac{\omega''}{2\pi}$ are always in phase with each other.

(*c*) Isochronous Circuits of Equal Damping.—This is another interesting special case. The conditions are

$$k_1 = k_2 = k_0 \tag{839}$$

$$p_1 = p_2 = p_0 \tag{840}$$

$$\omega_1 = \omega_2 = \omega_0. \tag{841}$$

Inserting these in Eq. (797),

$$(1 \pm \mu)(\omega j - k)^2 + 2k_0(\omega j - k) + (\omega_0{}^2 + k_0{}^2) = 0 \tag{842}$$

$$\therefore\quad (1 \pm \mu)(-\omega^2 + k^2) - 2kk_0 + (\omega_0{}^2 + k_0{}^2) = 0 \tag{843}$$

and
$$(1 \pm \mu)2\omega k - 2\omega k_0 = 0.$$

Therefore, $$k = \frac{k_0}{(1 \pm \mu)} \tag{844}$$

and $$(1 \pm \mu)(-\omega^2 + k^2) - 2k^2(1 \pm \mu) + (\omega_0^2 + k_0^2) = 0, \tag{845}$$

*i.e.*, $$\omega_0^2 + k_0^2 = (1 \pm \mu)(\omega^2 + k^2). \tag{846}$$

It is interesting to note that if

$$(\omega^2 + k^2) = p^2 \tag{847}$$

then $$p^2 = \frac{p_0^2}{(1 \pm \mu)}, \tag{848}$$

*i.e.*, $$p'^2 = \frac{p_0^2}{(1 - \mu)} \tag{849}$$

$$p''^2 = \frac{p_0^2}{(1 + \mu)}, \tag{850}$$

which can be compared with the corresponding result for $\omega'$ and $\omega''$ in terms of $\omega_0$ in isochronous circuits of negligible damping. The results are, however, more conveniently stated in the form

$$k' = \frac{k_0}{(1 - \mu)} \tag{851}$$

$$k'' = \frac{k_0}{(1 + \mu)} \tag{852}$$

$$\omega'^2 = \frac{p_0^2}{(1 - \mu)} - \frac{k_0^2}{(1 - \mu)^2} \tag{853}$$

$$\omega''^2 = \frac{p_0^2}{(1 + \mu)} - \frac{k_0^2}{(1 + \mu)^2}. \tag{854}$$

For the determination of a and b

$$a + jb = -\frac{L_1}{M}\frac{(\omega j - k)^2 + 2k_0(\omega j - k) + (\omega_0^2 + k_0^2)}{(\omega j - k)^2}. \tag{855}$$

Operating on the numerator and the denominator with $(\omega j + k)$ and putting $\omega^2 + k^2 = p^2$, $\omega_0^2 + k_0^2 = p_0^2$,

$$(a + jb) = -\frac{L_1\{p^4 - 2k_0p^2(\omega j + k) + p_0^2(\omega j + k)^2\}}{Mp^4} \tag{856}$$

$$a = -\frac{L_1}{M}\left\{1 - \frac{2kk_0}{p^2} + \frac{p_0^2}{p^4}(k^2 - \omega^2)\right\} \tag{857}$$

and $$b = \frac{L_1}{M}\left(2k_0\omega - \frac{p_0^2}{p^2}2k\omega\right). \tag{858}$$

Substituting in these expressions the values of p and k in terms of $p_0$ and $k_0$ derived from Eqs. (851) to (854)

$$a = \frac{-L_1}{M}\Big[1 - \frac{2k_0^2}{p^2(1 \mp \mu)} + \frac{(1 \mp \mu)}{p^2}$$

$$\left\{\frac{k_0^2}{(1 \mp \mu)^2} - \frac{p_0^2}{(1 \mp \mu)} + \frac{k_0^2}{(1 \mp \mu)^2}\right\}\Big] \qquad (859)$$

$$= \frac{-L_1}{M}\left(1 - \frac{p_0^2}{p^2}\right) \qquad (860)$$

$$= \frac{-L_1}{M}\left\{1 - (1 \mp \mu)\right\} \qquad (861)$$

$$\therefore \quad a' = \frac{-L_1}{M} \cdot \mu = -\sqrt{\frac{L_1}{L_2}} \text{ (\textit{cf.} Eq. (837))} \qquad (862)$$

$$a'' = \frac{-L_1}{M} \cdot -\mu = +\sqrt{\frac{L_1}{L_2}} \text{ (\textit{cf.} Eq. (838))} \qquad (863)$$

Also,
$$b' = 2k(1 \mp \mu)\omega - (1 \mp \mu)2k\omega \qquad (864)$$

$$= 0 \qquad (865)$$

$$\therefore \quad b' = b'' = 0. \qquad (866)$$

Thus, the interesting result is that the ratios of the corresponding oscillations in the two circuits are the same whether the damping is negligible or not, provided the damping is the same in the two circuits.

3. *The Amplitude and the Phase of the Oscillations.*—As in the free oscillation of a single oscillatory circuit, the amplitudes and the phases of the oscillations under consideration will depend on the nature of the disturbances which give rise to them. There are, in general, four unknown constants to be determined, $\hat{I}'$, $\hat{I}''$, $\psi'$, and $\psi''$. Four boundary conditions will, therefore, be required, the substitution of which in the general solution will give four equations to be solved simultaneously. In the general case this is a quite straightforward but rather lengthy process. The method will, therefore, be illustrated by the analysis of the simpler special case of isochronous circuits of equal damping, the condensers of which are charged to potentials $v_1$ and $v_2$ by means of some source of continuous potential in series with the circuits, which sources are removed instantaneously or short-circuited at the instant $t = 0$.

The boundary conditions in this case will, therefore, be

$$i_1 = 0 \quad \text{when } t = 0 \qquad (867)$$

$$i_2 = 0 \quad \text{when } t = 0 \qquad (868)$$

$$e_{c1} = v_1 \quad \text{when } t = 0 \qquad (869)$$

$$e_{c2} = v_2 \quad \text{when } t = 0 \qquad (870)$$

$e_{c1}$ and $e_{c2}$ being the potential differences across the condensers $C_1$ and $C_2$ respectively.

Stating the general solution in the form

$$i_1 = (I' + I'')\cdot \nu \tag{871}$$

$$i_2 = \{(a' + jb')I' + (a'' + jb'')I''\}\cdot \nu \tag{872}$$

where

$$I' \cdot \nu = \hat{I}'\epsilon^{-k't} \cos(\omega' t + \psi') \tag{873}$$

$$I'' \cdot \nu = \hat{I}''\epsilon^{-k''t} \cos(\omega' t + \psi'') \tag{874}$$

the results for the special case under consideration are

$$k' = \frac{k_0}{(1 - \mu)} \tag{875}$$

$$k'' = \frac{k_0}{(1 + \mu)} \tag{876}$$

$$p'^2 = \frac{p_0^2}{(1 - \mu)} \tag{877}$$

$$p''^2 = \frac{p_0^2}{(1 + \mu)} \tag{878}$$

where

$$p_0^2 = \omega_0^2 + k_0^2. \tag{879}$$

The general solution, therefore, takes the form

$$i_1 = (I' + I'') \cdot \nu \tag{880}$$

$$i_2 = a'(I' - I'') \cdot \nu. \tag{881}$$

For $e_{c1}$ the form is

$$e_{c1} = -\left\{\frac{I'}{C_1(\omega' j - k')} + \frac{I''}{C_1(\omega'' j - k'')}\right\} \cdot \nu \tag{882}$$

$$= \left\{\frac{\omega' j + k'}{C_1 p'^2} I' + \frac{\omega'' j + k''}{C_1 p''^2} I''\right\} \cdot \nu \tag{883}$$

$$= \frac{k_0}{C_1 p_0^2}(I' + I'') \cdot \nu + \left(\frac{\omega'}{C_1 p'^2} jI' + \frac{\omega''}{C_1 p''^2} jI''\right) \cdot \nu, \tag{884}$$

since

$$\frac{k'}{p'^2} = \frac{k''}{p''^2} = \frac{k_0}{p_0^2}. \tag{885}$$

Similarly, $e_{c2}$ is given by

$$e_{c2} = \frac{a' k_0}{C_2 p_0^2}(I' - I'') \cdot \nu + a'\left(\frac{\omega'}{C_2 p'^2} jI' - \frac{\omega''}{C_2 p''^2} jI''\right) \cdot \nu. \tag{886}$$

At the instant $t = 0$,

$$I' \cdot \nu = \hat{I}' \cos \psi' \tag{887}$$

$$I'' \cdot \nu = \hat{I}'' \cos \psi'' \tag{888}$$

$$jI' \cdot \nu = -\hat{I}' \sin \psi' \tag{889}$$

$$jI'' \cdot \nu = -\hat{I}'' \sin \psi''. \tag{890}$$

Inserting the first two boundary conditions in Eqs. (871) and (872),

$$\hat{I}' \cos \psi' + \hat{I}'' \cos \psi'' = 0 \tag{891}$$

$$a'(\hat{I}' \cos \psi' - \hat{I}'' \cos \psi'') = 0. \tag{892}$$

Therefore,

$$\hat{I}' \cos \psi' = \hat{I}'' \cos \psi'' = 0. \tag{893}$$

Inserting this result and the second two boundary conditions in Eqs. (884) and (886),

$$\frac{\omega'}{p'^2}\hat{I}' \sin \psi' + \frac{\omega''}{p''^2}\hat{I}'' \sin \psi'' = -C_1 v_1 \tag{894}$$

$$\frac{\omega'}{p'^2}\hat{I}' \sin \psi' - \frac{\omega''}{p''^2}\hat{I}'' \sin \psi'' = -C_2 v_2. \tag{895}$$

Therefore,

$$\hat{I}' \sin \psi' = -\frac{p'^2}{\omega'}\left(C_1 v_1 + \frac{C_2 v_2}{a'}\right) \tag{896}$$

$$\hat{I}'' \sin \psi'' = -\frac{p''^2}{\omega''}\left(C_1 v_1 - \frac{C_2 v_2}{a'}\right). \tag{897}$$

The complete solution, therefore, takes the form

$$i = \frac{p'^2}{\omega'}\left(C_1 v_1 + \frac{C_2 v_2}{a'}\right)\epsilon^{-k't} \sin \omega' t + \frac{p''^2}{\omega''}\left(C_1 v_1 - \frac{C_2 v_2}{a'}\right)\epsilon^{-k''t} \sin \omega'' t \tag{898}$$

$$i = \frac{a'p'^2}{\omega'}\left(C_1 v_1 + \frac{C_2 v_2}{a'}\right)\epsilon^{-k't} \sin \omega' t - \frac{a'p''^2}{\omega''}\left(C_1 v_1 - \frac{C_2 v_2}{a'}\right)\epsilon^{-k''t} \sin \omega'' t, \tag{899}$$

where, repeating the abbreviations for convenience,

$$a' = -\sqrt{\frac{L_1}{L_2}} \tag{900}$$

$$\omega'^2 = \frac{p_0^2}{(1-\mu)} - \frac{k_0^2}{(1-\mu)^2} \tag{901}$$

$$\omega''^2 = \frac{p_0^2}{(1+\mu)} - \frac{k_0^2}{(1+\mu)^2} \tag{902}$$

$$p'^2 = \frac{p_0^2}{(1-\mu)} \tag{903}$$

$$p''^2 = \frac{p_0^2}{(1+\mu)} \tag{904}$$

$$k' = \frac{k_0}{(1-\mu)} \tag{905}$$

$$k'' = \frac{k_0}{(1+\mu)} \tag{906}$$

where $$p_0^2 = \frac{1}{L_1C_1} = \frac{1}{L_2C_2} \tag{907}$$

$$k_0 = \frac{R_1}{2L_1} = \frac{R_2}{2L_2} \tag{908}$$

and $$\mu^2 = \frac{M^2}{L_1L_2}. \tag{909}$$

(*a*) The Conditions for Oscillations of One Frequency Only.—In each circuit the amplitude of the oscillation of the higher frequency, corresponding to $\omega'$, is proportional to

$$C_1v_1 - C_2v_2\sqrt{\frac{L_2}{L_1}}.$$

Similarly, the amplitude of the lower-frequency oscillation is proportional to

$$C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}}.$$

Now $v_1$ and $v_2$ may be of the same or of opposite signs. It should be noted that the sign convention implied throughout the whole of this analysis is that currents of the same sign will produce magnetic fluxes of the same direction in the two inductances. It is for this reason that the oscillations which are in phase with each other correspond to the lower frequency. The potentials $v_1$ and $v_2$ will be reckoned to be of the same or of opposite sign in accordance with this same convention. If they are of the same sign, then clearly the amplitude of the higher frequency oscillation may be reduced to zero by making

$$C_1v_1 = C_2v_2\sqrt{\frac{L_2}{L_1}}. \tag{910}$$

This condition is equivalent to

$$\sqrt{L_1C_1} \cdot \sqrt{C_1}\,v_1 = \sqrt{L_2C_2} \cdot \sqrt{C_2}\,v_2 \tag{911}$$

or $$C_1v_1^2 = C_2v_2^2. \tag{912}$$

Thus, if the condensers are charged in the same direction with equal quantities of energy, only one oscillation will be produced, that having the lower frequency.

Similarly, if the condensers are charged in opposite directions with the same amount of energy,

$$C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}} = 0 \tag{913}$$

and only the higher-frequency oscillation will result.

(*b*) An interesting dynamical analogy of the electrical system under consideration is the well-known case of the sympathetic pendulums, illustrated in Fig. 67.

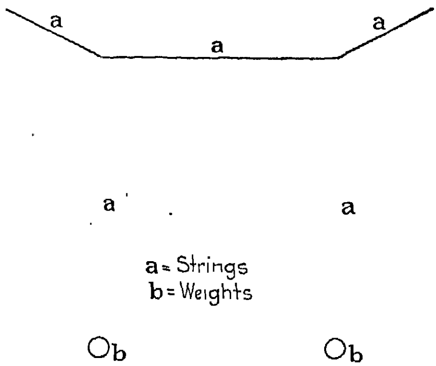


FIG. 67.

If both pendulums are displaced to the same extent in the same direction and then let go, they will continue to swing in unison with a single-constant frequency, their amplitudes diminishing together, owing to the dissipative frictional forces of the system. This corresponds to the case in which the condensers of the electrical system are charged with equal quantities of energy in the same direction. If the pendulums are displaced an equal extent in opposite directions, they will continue to swing in antiphase with each other with a single-constant frequency. This corresponds to the case in which the condensers are charged with opposite potentials. Moreover, with the pendulums it will be found that the frequency in the second case is higher than that in the first, which, again, is in agreement with the electrical analogy.

If, now, the pendulums are displaced in either the same or opposite directions with unequal amplitudes, or if only one pendulum is displaced, the succession of events will be quite different. It will be found that the swing of the pendulum which was given the larger displacement will gradually diminish, while the amplitude of the other pendulum gradually increases until the former is apparently stationary or at least is swinging with a small amplitude, while the latter is swinging vigorously. Then the stationary one will commence to build up an oscillation, while the movement of the other dies down until the conditions of the two have been reversed. The process will continue till both pendulums have come to rest, owing to the mechanical decrements of the oscillations.

It will be interesting to see if this process can also be deduced from the equations of the electrical analogy. For this purpose it will be convenient to simplify the expressions still further by assuming the damping to be negligible, in which case, since $p' = \omega'$ and $p'' = \omega''$,

$$i_1 = p'\left(C_1v_1 - C_2v_2\sqrt{\frac{L_2}{L_1}}\right)\sin p't + p''\left(C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}}\right)\sin p''t \qquad (914)$$

$$i_2 = -\sqrt{\frac{L_1}{L_2}}\left\{ p'\left(C_1v_1 - C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p't \right.$$
$$\left. - p''\left( C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p''t\right\} \qquad (915)$$

or, in terms of $p_0$ and $\mu$,

$$i_1 = \frac{p}{\sqrt{1-\mu}}\left(C_1v_1 - C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p_0\frac{t}{\sqrt{1-\mu}}$$
$$+ \frac{p_0}{\sqrt{1+\mu}}\left(C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p_0\frac{t}{\sqrt{1+\mu}} \qquad (916)$$

$$i_2 = -\sqrt{\frac{L_1}{L_2}}\left\{ \frac{p}{\sqrt{1-\mu}}\left(C_1v_1 - C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p_0\frac{t}{\sqrt{1-\mu}}\right.$$
$$\left. + \frac{p_0}{\sqrt{1+\mu}}\left(C_1v_1 + C_2v_2\sqrt{\frac{L_2}{L_1}}\right) \sin p_0\frac{t}{\sqrt{1+\mu}}\right\}. \qquad (917)$$

The oscillations of the first circuit can be written

$$i_1 = A \sin p't + B \sin p''t \qquad (918)$$
$$= A (\sin p't + \sin p''t) + (B - A) \sin p''t \qquad (919)$$
$$= 2A \cos (p' - p'')\frac{t}{2} \sin (p' + p'')\frac{t}{2} + (B - A) \sin p''t. \qquad (920)$$

Thus the oscillations can be considered to consist of a constant term $(B - A) \sin p''t$, the amplitude of which will, in general, be small and will actually be zero if $B = A$, *i.e.*, if $v_2 = 0$, together with an oscillation of frequency $\frac{p' + p''}{2} \cdot \frac{1}{2\pi}$, the amplitude of which is $2A \cos \frac{p' - p''}{2}t$, *i.e.*, the amplitude fluctuates between 2A and zero $\frac{p' - p''}{2\pi}$ times a second.

In a precisely similar manner the oscillations in the second circuit can be represented by

$$i_2 = 2A \sin\frac{p' - p''}{2} t \cos\frac{p' + p''}{2}t - (B - A) \sin p''t, \qquad (921)$$

which are seen to be of exactly the same character as those in the first circuit. Moreover, when the amplitude of the main oscillation is zero in the first circuit it is a maximum in the second, and *vice versa*. The behavior of the electrical system is, in fact, strictly analogous with that of the sympathetic pendulums.

This phenomenon is well known in wireless stations transmitting damped wave trains. In these cases the aerial constitutes the second circuit, damped wave trains being generated in another circuit to which it is coupled inductively. If this coupling is made too close, energy surges backwards and forwards between the aerial and the generating circuit, and two wave lengths are transmitted by the aerial. It is for this reason that the quenched-spark system was introduced, the object being to prevent the return of energy to the generating circuit from the aerial, at the same time making use of the advantages of a fairly close coupling.

## EXAMPLES

**1.** Draw the locus of the end of the vector representing the damped electric oscillation

$$i = 10\epsilon^{-t/5} \cos \pi t.$$

**2.** A damped electric oscillation is represented by the vector

$$I = I_0\epsilon^{-kt}$$

rotating with constant angular velocity $\omega$. Find the ratio between the vectors representing the oscillation

(*a*) At any two instants separated by a time interval $\frac{T}{2}$, *i.e.*, one-half of a period.

(*b*) At any two instants separated by a time interval T.

**3.** A damped electric oscillation

$$i = 10^{-3}\epsilon^{-10t} \cos 100t \text{ amperes}$$

represented at the instant $t = 0$ by the vector $10^{-3}$ v flows through:

(*a*) A pure resistance of 10 ohms.

(*b*) A pure inductance of 1 millihenry.

(*c*) A pure capacity of 5 microfarads.

Find the expressions for the back e.m.fs. produced across the conductors (1) in vector form, at the instant $t = 0$, (2) in scalar form at time t.

**4.** Express in scalar form the instantaneous value of the rate of change of energy associated with the three conductors specified in Example 3.

**5.** An e.m.f. represented by

$$e = \hat{E}_0\epsilon^{-kt} \cos\left(\omega t + \frac{\pi}{2}\right)$$

is applied to the plates of a condenser of capacity C.

(*a*) Find the scalar expression for the instantaneous value of the rate of change of energy associated with the condenser.

(*b*) Show, by integration, that the total change of energy from $t = 0$ to $t = \infty$ is zero.

**6.** A closed circuit consists of a pure resistance of 10 ohms in series with a pure inductance of 5 millihenries in series with a pure capacity of 1,000 micro-microfarads. A continuous potential difference of 10 volts is maintained across the plates of the condenser. Find the complete scalar expression for the free oscillation which takes place when the source of continuous potential difference is suddenly removed.

## ANSWERS TO EXAMPLES

**2.** (*a*) $\dfrac{I_t}{I_{t+\frac{T}{2}}} = -\epsilon^{k\frac{T}{2}}.$

(*b*) $\dfrac{I_t}{I_{t+T}} = \epsilon^{kT}.$

**3.** (1) (*a*) $-10^{-2}$ v.

(*b*) $100.5 \times 10^{-6}\epsilon^{-j84°18'}$ v.

(*c*) $1.99\epsilon^{j84°18'}$ v.

(2) (*a*) $-10^{-2}\epsilon^{-10t}\cos 100t.$

(*b*) $100.5 \times 10^{-6}\epsilon^{-10t}\cos(100t - 84°\ 18').$

(*c*) $1.99\epsilon^{-10t}\cos(100t + 84°\ 18').$

**4.** (*a*) $-5 \times 10^{-6}\epsilon^{-20t}(1 - \cos 200t)$ watts.

(*b*) $5.025 \times 10^{-8}\epsilon^{-20t}\{.099 + \cos(200t - 84°\ 18')\}$ watts.

(*c*) $.995 \times 10^{-3}\epsilon^{-20t}\{.099 + \cos(200t + 84°\ 18')\}$ watts.

**5.** (*a*) $-\frac{kC\hat{E}_0^2}{2}\epsilon^{-2kt} + \frac{kC\hat{E}_0^2}{2}\epsilon^{-2kt}\cos 2\omega t + \frac{\omega C\hat{E}_0^2}{2}\epsilon^{-2kt}\sin 2\omega t.$

(*b*) Integration.

$$\frac{C\hat{E}_0^2}{2}\left\{\frac{-k}{2k} + \frac{k^2}{2(\omega^2 + k^2)} + \frac{\omega^2}{2(\omega^2 + k^2)}\right\} = 0.$$

**6.** $\epsilon^{-10^5 t}(\cos 4.47 \times 10^6 t + .0021 \sin 4.47 \times 10^6 t).$

## CHAPTER VII

# THE APPLICATION OF VECTOR ANALYSIS TO THE THEORY OF ALTERNATING CURRENTS OF IRREGULAR WAVE SHAPE

**64. Alternating Currents in Practice.**—Up to this point only those alternating currents of constant amplitude have been considered which can be represented in scalar form by an equation of the type

$$i = \hat{I} \cos (\omega t + \theta) \tag{922}$$

and in vector form by a vector **I** of constant magnitude $\hat{I}$ rotating with uniform angular velocity $\omega$ relative to a fixed unit vector of reference $\mathbf{v}$.

Alternating currents of this simple type are the ideal of alternating-current engineering, and modern practice tends more and more to their realization. It is, nevertheless, probably true that this ideal wave form is still the exception rather than the rule. Even where e.m.fs. of pure sine-wave form are obtained, certain conditions, for instance, inductive loads including magnetic circuits in iron, will result in a distorted wave form for the currents produced by them.

For the theoretical discussion of alternating currents and potentials the wave forms of which differ appreciably from that of the simple sine curve, the methods already described will prove inadequate, and, in order to deal with such cases, it will be necessary to reconsider them in relation to the general type of alternating current represented in scalar form by an equation of the type

$$i = f(t), \tag{923}$$

where $f(t)$ is any finite and single-valued function of time of a periodic character, *i.e.*, $f(t)$ is such that for all values of t

$$f(t) = f(t + T), \tag{924}$$

T being the period of the function.

For the present discussion it will be preferable to restrict the analysis to currents which are truly alternating, in the sense that

they contain no component which is not of a periodic character, *i.e.*, no constant component. This characteristic is expressed mathematically by the condition that the average value of the function over a period is zero, *i.e.*,

$$\frac{1}{T}\int_0^T f(t)dt = 0. \tag{925}$$

A typical example of such a function is illustrated graphically in Fig. 68.

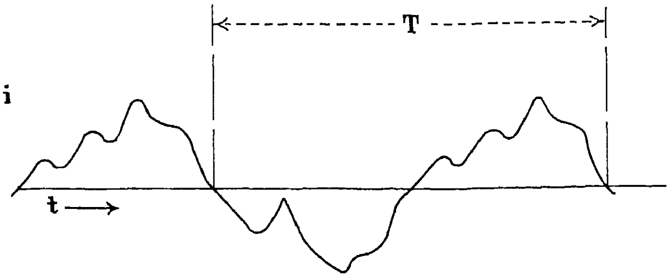


FIG. 68.

**65. The Vectorial Representation of the General Type of Alternating Current.**—The simplifications introduced into the theoretical discussion of alternating currents of pure sine-wave form by the use of vectors depend largely on the simple character of the vectorial expressions involved. In dealing with alternating currents of general wave shape, this simplicity is no longer available. It is, in fact, arguable whether the vectorial method has, in the general case, any great advantage over the ordinary scalar analysis. It is, however, at least as effective as the latter and for the student who has familiarized himself with the vectorial method it will have the advantage of bringing the general case into line with the simple special case of the pure sine wave. It has, moreover, the great advantage of compactness, a characteristic which will prove of considerable value in the discussion of polyphase systems of general wave form.

The vectorial representation of alternating currents of general wave form depends on Fourier's theorem, which, so far as the present subject is concerned, states that any periodic function of the type described in Par. 64 can be expressed as the sum of an infinite series of simple cosine terms as shown in Eq. (926).

$$i = f(t) = \hat{I}_1 \cos(\omega t + \theta_1) + \hat{I}_2 \cos(2\omega t + \theta_2) + \hat{I}_3 \cos(3\omega t + \theta_3) + \ldots + \hat{I}_n \cos(n\omega t + \theta_n) + \ldots \text{ etc., etc. } \textit{ad inf.} \tag{926}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n \cos(n\omega t + \theta_n) \tag{927}$$

where

$$\omega = \frac{2\pi}{T}. \tag{928}$$

In the above, $\hat{I}_n \cos(n\omega t + \theta_n)$ is known as the $n^{th}$ harmonic, $\hat{I}$ being its amplitude and $\theta_n$ its phase at the instant $t = 0$.

The first term $\hat{I}_1 \cos(\omega t + \theta_1)$ is known as the fundamental.

Each of the individual members of the above series can be regarded as an alternating current of pure sine-wave form, and can, therefore, be represented by a rotating vector $\mathbf{I}_n$ of constant magnitude $\hat{I}_n$ and uniform angular velocity $n\omega$, making an angle $\theta_n$ with the unit vector $\boldsymbol{\nu}$ at the instant $t = 0$, *i.e.*,

$$i_n = \mathbf{I}_n \cdot \boldsymbol{\nu} = \hat{I}_n \cos(n\omega t + \theta_n) \tag{929}$$

so that Eq. (926) may be written:

$$i = \sum_{n=1}^{n=\infty} i_n = \mathbf{I}_1 \cdot \boldsymbol{\nu} + \mathbf{I}_2 \cdot \boldsymbol{\nu} + \mathbf{I}_3 \cdot \boldsymbol{\nu} + \ldots + \mathbf{I}_n \cdot \boldsymbol{\nu} + \ldots \text{ etc., etc. } \textit{ad inf.} \tag{930}$$

$$= (\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \ldots + \mathbf{I}_n + \ldots \text{ etc., etc. } \textit{ad inf.}) \cdot \boldsymbol{\nu} \tag{931}$$

$$= \mathbf{I} \cdot \boldsymbol{\nu} \tag{932}$$

where

$$\mathbf{I} = \sum_{n=1}^{n=\infty} \mathbf{I}_n. \tag{933}$$

Thus the alternating current of instantaneous value $i$ can be represented as the scalar product with $\boldsymbol{\nu}$ of the vector $\mathbf{I}$ defined in accordance with Eqs. (929) to (933).

It is clear that the vector $\mathbf{I}$ so defined will not in general be uniform in angular velocity nor constant in magnitude. Its magnitude and velocity will, however, be periodic functions of time, so that it can be described as a periodic vector.

*Fourier's theorem has thus a vectorial form of expression, in which the periodic vector* $\mathbf{I}$ *is represented as the sum of an infinite number of rotating vectors of constant magnitude and uniform angular velocity.*

**66. Vectorial Expression for the Magnitudes of the Harmonics.**—Let $\boldsymbol{\nu}_n$ be a unit vector coincident with $\boldsymbol{\nu}$ at the instant $t =$

0, and rotating with uniform angular velocity $n\omega$, n being any positive integer. Then, if $\mathbf{I}_m$ be the vector representing the $m^{th}$ harmonic of **I**,

$$\mathbf{I}_m \cdot \mathbf{v}_n = \hat{I}_m \cos\{(m-n)\omega t + \theta_m\} \qquad (934)$$

and, since $(m - n)$ is a whole number,

$$\frac{1}{T}\int_0^T (\mathbf{I}_m \cdot \mathbf{v}_n)dt = \frac{1}{T}\int_0^T \hat{I}_m \cos\{(m-n)\omega t + \theta_m\}dt. \qquad (935)$$
$$= 0.$$

On the other hand, for the vector $\mathbf{I}_n$,

$$\mathbf{I}_n \cdot \mathbf{v}_n = \hat{I}_n \cos \theta_n, \qquad (936)$$

therefore

$$\frac{1}{T}\int_0^T (\mathbf{I}_n \cdot \mathbf{v}_n)dt = \hat{I}_n \cos\theta_n. \qquad (937)$$

Considering now the equation

$$\mathbf{I} = \sum_{n=1}^{n=\infty} \mathbf{I}_n, \qquad (938)$$

the result of taking the scalar product of both sides with $\mathbf{v}_n$ is

$$\mathbf{I} \cdot \mathbf{v}_n = \left(\sum_{n=1}^{n=\infty} \mathbf{I}_n\right) \cdot \mathbf{v}_n;$$

therefore,

$$\frac{1}{T}\int_0^T (\mathbf{I} \cdot \mathbf{v}_n)dt = \frac{1}{T}\int_0^T \left(\sum_{n=1}^{n=\infty} \mathbf{I}_n\right) \cdot \mathbf{v}_n dt \qquad (939)$$
$$= \hat{I}_n \cos\theta_n,$$

since, as shown above, the whole of the remaining terms vanish.

Similarly, by taking the scalar product with $j\mathbf{v}_n$,

$$\frac{1}{T}\int_0^T (\mathbf{I} \cdot j\mathbf{v}_n)dt = \hat{I}_n \sin\theta_n. \qquad (940)$$

From the two equations

$$\hat{I}_n \cos\theta_n = \frac{1}{T}\int_0^T (\mathbf{I} \cdot \mathbf{v}_n)dt \qquad (941)$$

$$\hat{I}_n \sin\theta_n = \frac{1}{T}\int_0^T (\mathbf{I} \cdot j\mathbf{v}_n)dt \qquad (942)$$

it is, of course, a simple matter to determine both $\hat{I}_n$ and $\theta_n$, *i.e.*, to determine the vector $\mathbf{I}_n$ completely.

It is of interest to compare the foregoing expressions with the corresponding results in scalar terms, *i.e.*, with the well-known equations:

$$\hat{I}_n \cos\theta_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t \, dt \qquad (943)$$

$$\hat{I}_n \sin\theta_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t \, dt. \qquad (944)$$

If I be of magnitude $\hat{I}$ and slope $\theta$

$$f(t) = \mathbf{I} \cdot \nu = \hat{I} \cos \theta \tag{945}$$

also

$$\mathbf{I} \cdot \nu_n = \hat{I} \cos (\theta - n\omega t) \tag{946}$$

$$= \hat{I} \cos \theta \cos n\omega t + \hat{I} \sin \theta \sin n\omega t. \tag{947}$$

Now, since

$$\mathbf{I} \cdot \nu = \hat{I} \cos \theta \tag{948}$$

$$= \sum_{n=1}^{n=\infty} \mathbf{I}_n \cdot \nu \tag{949}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n), \tag{950}$$

then

$$\hat{I} \sin \theta = \mathbf{I} \cdot j\nu.$$

Therefore,

$$\hat{I} \sin \theta = \sum_{n=1}^{n=\infty} (\mathbf{I}_n \cdot j\nu) \tag{951}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n \sin (n\omega t + \theta_n), \tag{952}$$

so that

$$\frac{1}{T}\int_0^T (\mathbf{I} \cdot \nu_n)dt = \frac{1}{T}\int_0^T (\hat{I} \cos \theta \cos n\omega t)dt + \frac{1}{T}\int_0^T (\hat{I} \sin \theta \sin n\omega t)dt \tag{953}$$

$$= \frac{1}{T}\int_0^T \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n) \right\} \cos n\omega t \, dt + \frac{1}{T}\int^T \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \sin (n\omega t + \theta_n) \right\} \sin n\omega t \, dt. \tag{954}$$

Now it is easy to verify that

$$\frac{1}{T}\int_0^T \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n) \right\} \cos n\omega t \, dt = \frac{1}{T}\int_0^T \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \sin (n\omega t + \theta_n) \right\} \sin n\omega t \, dt. \tag{955}$$

Therefore, finally,

$$\frac{1}{T}\int_0^T (\mathbf{I} \cdot \nu_n)dt = \frac{2}{T}\int_0^T \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n) \right\} \cos n\omega t \, dt \tag{956}$$

$$= \frac{2}{T}\int_0^T \{f(t)\} \cos n\omega t \, dt, \tag{957}$$

so that the expression for $\hat{I}_n \cos \theta_n$ in vectorial terms is proved to be identical with its usual form of expression in scalar terms.

**67. General Alternating Currents Containing Odd Harmonics Only.**—In a half a period the vector $\mathbf{I}_1$ which represents the fundamental will rotate through 180°. Similarly, the vector $\mathbf{I}_n$

which represents the $n^{th}$ harmonic will in the same time rotate through n times 180°, or $n\pi$ radians. In other words, if the $n^{th}$ harmonic be represented at the instant t by the vector $\mathbf{I}_n$, it will be represented at the instant $\left(t + \frac{T}{2}\right)$ by the vector $(-1)^n\mathbf{I}_n$.

Thus, if

$$\mathbf{I}_{(t)} = (\mathbf{I}_1 + \mathbf{I}_3 + \mathbf{I}_5 + \text{etc.}) + (\mathbf{I}_2 + \mathbf{I}_4 + \mathbf{I}_6 + \text{etc.}) \quad (958)$$

be the vector **I** at the instant t, at the instant $\left(t + \frac{T}{2}\right)$

$$\mathbf{I}_{\left(t+\frac{T}{2}\right)} = -(\mathbf{I}_1 + \mathbf{I}_3 + \mathbf{I}_5 + \text{etc.}) + (\mathbf{I}_2 + \mathbf{I}_4 + \mathbf{I}_6 + \text{etc.}) \quad (959)$$

Therefore,

$$\mathbf{I}_{(t)} + \mathbf{I}_{\left(t+\frac{T}{2}\right)} = 2(\mathbf{I}_2 + \mathbf{I}_4 + \mathbf{I}_6 + \mathbf{I}_8 + \text{etc.}). \quad (960)$$

If, therefore, the wave contains no even harmonics,

$$\mathbf{I}_{(t)} = -\mathbf{I}_{\left(t+\frac{T}{2}\right)} \quad (961)$$

and

$$\mathbf{I}_{(t)} \cdot \boldsymbol{\nu} = -\mathbf{I}_{\left(t+\frac{T}{2}\right)} \cdot \boldsymbol{\nu} \quad (962)$$

or, in scalar form,

$$f(t) = -f\left(t + \frac{T}{2}\right). \quad (963)$$

A function of this type is illustrated in Fig. 69. In practice, the majority of alternating currents are of this type, and contain

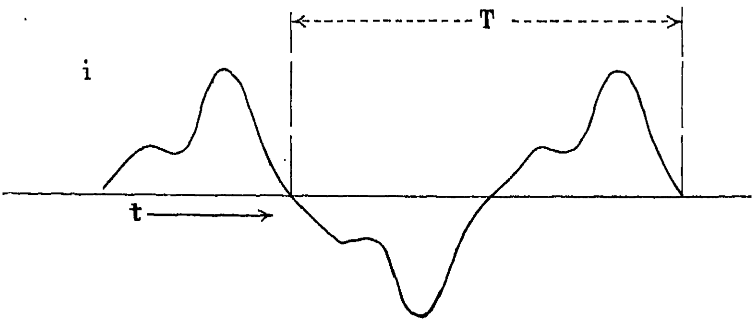


Fig. 69.

only odd harmonics, since this is the type of wave produced by ordinary generators. Currents containing even harmonics as well are, however, occasionally met, as, for instance, in some cases of arc or valve generation.

**68. The Application of Kirchhoff's Laws to Vectors Representing Alternating Currents and Potentials of General Wave Form.**—It was shown in Par. 34 that Kirchhoff's first and second laws could be applied to vectors representing alternating currents and potentials of pure sine-wave form. The proof given required only that the vectors correctly represented by their scalar products with the fixed unit vector of reference, $\mathbf{v}$, the instantaneous values of the currents and potentials concerned. Since this condition is fulfilled by the vectors which represent alternating currents and potentials of general wave form, the same proof will apply, and will, therefore, not be restated here. In the present case, however, a further deduction can be made which will prove of considerable practical importance.

Consider the case in which r conductors, numbered 1 to r for reference, meet at a point in a network, the currents in the conductors being represented by the vectors $\mathbf{I}_{(1)}$, $\mathbf{I}_{(2)}$, $\mathbf{I}_{(3)}$, etc., etc., up to $\mathbf{I}_{(r)}$.

Applying Kirchhoff's first law to the point,

$$\mathbf{I}_{(1)} + \mathbf{I}_{(2)} + \mathbf{I}_{(3)} + \ . \ . \ . \ + \mathbf{I}_{(r)} = 0. \qquad (964)$$

Assuming each of these vectors to represent a current of general wave form, then

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \mathbf{I}_{(1)n} \qquad (965)$$

$$\mathbf{I}_{(2)} = \sum_{n-1}^{n=\infty} \mathbf{I}_{(2)n} \qquad (966)$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \mathbf{I}_{(3)n} \qquad (967)$$

etc., etc.

Equation (964) can, therefore, be written

$$\sum_{n=1}^{n=\infty} \mathbf{I}_{(0)n} + \sum_{n=1}^{n=\infty} \mathbf{I}_{(2)n} + \sum_{n=1}^{n=\infty} \mathbf{I}_{(3)n} + \ . \ . \ . \ + \sum_{n=1}^{n=\infty} \mathbf{I}_{(r)n} = 0. \qquad (968)$$

Now let

$$\mathbf{I}_{(s)n} = \mathbf{I}_{(1)n} + \mathbf{I}_{(2)n} + \mathbf{I}_{(3)n} + \ . \ . \ . \ + \mathbf{I}_{(r)n}, \qquad (969)$$

*i.e.*, $\mathbf{I}_{(s)n}$ is the sum of all the currents of frequency $\frac{n\omega}{2\pi}$, *i.e.*, of all the $n^{th}$ harmonics of the various currents.

Equation (964) becomes

$$\mathbf{I}_{(s)1} + \mathbf{I}_{(s)2} + \mathbf{I}_{(s)3} + \dots + \mathbf{I}_{(s)n} + \dots = \sum \mathbf{I}_{(s)n} \quad (970)$$
$$= 0. \quad (971)$$

Taking the scalar product of Eq. (970) with $\boldsymbol{\nu}_n$, where $\boldsymbol{\nu}_n$ is a unit vector coincident with $\boldsymbol{\nu}$ at the instant $t = 0$, and rotating with uniform angular velocity $n\omega$, then

$$\mathbf{I}_{(s)1} \cdot \boldsymbol{\nu}_n + \mathbf{I}_{(s)2} \cdot \boldsymbol{\nu}_n + \mathbf{I}_{(s)3} \cdot \boldsymbol{\nu}_n + \dots + \mathbf{I}_{(s)n} \cdot \boldsymbol{\nu}_n + \dots = 0. \quad (972)$$

Therefore,

$$\frac{1}{T}\int_0^T (\mathbf{I}_{(s)1} \cdot \boldsymbol{\nu}_n)dt + \frac{1}{T}\int_0^T (\mathbf{I}_{(s)2} \cdot \boldsymbol{\nu}_n)dt + \frac{1}{T}\int^T (\mathbf{T}_{(s)3} \cdot \boldsymbol{\nu}_n)dt \dots \frac{1}{T}\int_0^T (\mathbf{I}_{(s)n} \cdot \boldsymbol{\nu}_n)dt + \dots = 0. \quad (973)$$

But, as shown in Par. 67, the whole of the above integrals vanish separately except $\frac{1}{T}\int_0^T (\mathbf{I}_{(s)n} \cdot \boldsymbol{\nu}_n)dt$. If

$$\mathbf{I}_{(s)n} \cdot \nu = \hat{I}_{(s)n} \cos (n\omega t + \theta_{(s)n}) \quad (974)$$

then for this integral

$$\frac{1}{T}\int_0^T (\mathbf{I}_{(s)n} \cdot \boldsymbol{\nu}_n)dt = \hat{I}_{(s)n} \cos \theta_{(s)n}. \quad (975)$$

Therefore, since the whole series equals zero, and all the integrals vanish separately except this one,

$$\hat{I}_{(s)n} \cos \theta_{(s)n} = 0. \quad (976)$$

Similarly, by taking the scalar product with $j\boldsymbol{\nu}_n$ it can be shown that

$$\hat{I}_{(s)n} \sin \theta_{(s)n} = 0. \quad (977)$$

*i.e.*

$$\hat{I}_{(s)n} = 0 \quad (978)$$

and $\theta_{(s)n}$ is indeterminate. Therefore,

$$\mathbf{I}_{(s)n} = 0. \quad (979)$$

Since n is any positive integer, this result holds for all values of n, *i.e.*, for all the harmonics. Thus, if

$$\mathbf{I}_{(s)1} + \mathbf{I}_{(s)2} + \mathbf{I}_{(s)3} + \dots + \mathbf{I}_{(s)n} + \dots = 0, \quad (980)$$

then

$$\mathbf{I}_{(s)1} = \mathbf{I}_{(s)2} = \mathbf{I}_{(s)3} = \dots = \mathbf{I}_{(s)n} = 0. \quad (981)$$

In other words, if

$$\mathbf{I}_{(1)} + \mathbf{I}_{(2)} + \mathbf{I}_{(3)} + \ . \ . \ . \ + \mathbf{I}_{(r)}$$

$$= \sum_{n=1}^{n=\infty} \mathbf{I}_{(1)n} + \sum_{n=1}^{n=\infty} \mathbf{I}_{(2)n} + \sum_{n=1}^{n=\infty} \mathbf{I}_{(3)n} + \ . \ . \ . \ + \sum \mathbf{I}_{(r)n} \qquad (982)$$

$$= 0. \qquad (983)$$

Then $$\mathbf{I}_{(1)n} + \mathbf{I}_{(2)n} + \mathbf{I}_{(3)n} + \ . \ . \ . \ + \mathbf{I}_{(r)n} = 0 \qquad (984)$$

for all values of n.

*Thus Kirchhoff's first law can be applied not only to the total currents* $\mathbf{I}_{(1)}$, $\mathbf{I}_{(2)}$, $\mathbf{I}_{(3)}$, *etc., but it also applies to each group of currents of the same frequency considered separately.*

By a process exactly similar to the above it can be shown that the same result holds good for Kirchhoff's second law.

*It appears, therefore, that in networks carrying alternating currents of general wave form each harmonic behaves as if it were entirely alone and can be considered separately. The total results can then be obtained by the summation of all the separate harmonics.*

**69. General Wave Form Impedances.**—1. Considering the circuit shown in Fig. 29 and using the same notation as in Par. 35:

$$\mathbf{E} + \mathbf{E}_{(R)} = 0. \qquad (985)$$

Therefore, from Par. 68

$$\mathbf{E}_n + \mathbf{E}_{(R)n} = 0 \qquad (986)$$

and, since $$\mathbf{E}_{(R)n} = -R\mathbf{I}_{(n)}, \qquad (987)$$

then $$\mathbf{E}_n = R\mathbf{I}_n. \qquad (988)$$

Therefore, $$\mathbf{E} = R\mathbf{I} \qquad (989)$$

or $$\mathbf{I} = \frac{\mathbf{E}}{R}. \qquad (990)$$

For a pure resistance, therefore, the current vector is a simple multiple $\frac{1}{R}$ times the potential vector and is, therefore, of the same character. In other words, the current can be expressed in terms of the e.m.f. taken as a whole, and is of the same wave form.

2. *Inductance.*—Referring to Fig. 31,

$$\mathbf{E}_{(n)} + \mathbf{E}_{(L)n} = 0 \qquad (991)$$

and, since $$\mathbf{E}_{(L)n} = -jn\omega L\mathbf{I}_n, \qquad (992)$$

then $$\mathbf{E}_n = jn\omega L\mathbf{I}_n. \qquad (993)$$

Therefore,
$$I_n = \frac{1}{jn\omega L} E_n \tag{994}$$

and
$$I = \sum_{n=1}^{n=\infty} \frac{1}{jn\omega L} E_n. \tag{995}$$

In this case, therefore, **I** and **E** will not be of the same wave shape, and **I** cannot be expressed as the result of any simple operation on **E**. It should be noted that

$$\frac{I_n}{I_1} = \frac{j\omega L}{jn\omega L}\frac{E_n}{E_1} = \frac{1}{n}\frac{E_n}{E_1}. \tag{996}$$

Thus the ratio between the nth harmonic of the current and the fundamental is only $\frac{1}{nth}$ of the ratio of the corresponding terms of the e.m.f., *i.e.*, the wave form of the current will be much less distorted than that of the e.m.f.

3. *Capacity.*—Referring to Fig. 33, it can be shown, in a similar manner to the above, that

$$I = \sum_{n=1}^{n=\infty} jn\omega C\, E_n, \tag{997}$$

and, as in the case of the inductance, the current cannot be expressed in terms of **E**.

It should be noted for comparison with the inductive circuit of Fig. 31 that

$$\frac{I_n}{I_1} = \frac{jn\omega C}{j\omega C}\frac{E_n}{E_1} = n\frac{E_n}{E_1}, \tag{998}$$

so that the ratio of the nth harmonic of the current to the fundamental is n times that of the corresponding terms of the e.m.f. In other words, the current through a capacity load will be of much more distorted wave form than the e.m.f.

4. *General Impedance.*—In general, for the circuit shown in Fig. 70, in which $z$ is an impedance whose value at a frequency $\frac{n\omega}{2\pi}$, *i.e.*, at the frequency of the nth harmonic, is $(R_n + jX_n)$.

$$I = \sum_{n=1}^{n=\infty} \frac{1}{R_n + jX_n} E_n. \tag{999}$$

Taking as an example

$$X_1 = \omega L - \frac{1}{\omega C} \tag{1000}$$

$$X_n = n\omega L - \frac{1}{n\omega C}, \tag{1001}$$

if for the nth harmonic

$$n\omega L - \frac{1}{n\omega C} = 0, \tag{1002}$$

*i.e.,*

$$\omega^2 LC = \frac{1}{n^2}, \tag{1003}$$

then

$$z_n = R_n \tag{1004}$$

and

$$I_n = \frac{E_n}{R_n}. \tag{1005}$$

If $R_n$ is small, this harmonic may reach a considerable value. In alternating currents of general wave form there is always the possibility that one or more of the harmonics may be magnified by resonance in this way.

**70. Power Equations with the General Wave Form.**—Consider the case illustrated in Fig. 70, where $z$ is a general impedance

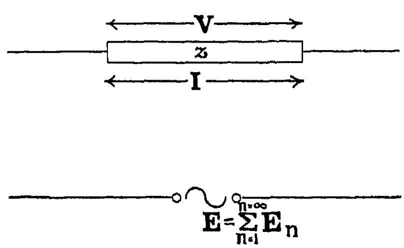


FIG. 70.

of the type $(R_n + jX)$, its value corresponding to the frequency $\frac{n\omega}{2\pi}$ being $(R_n + jX_n)$. If

$$i = \sum_{n=1}^{n=\infty} i_n = \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n) \tag{1006}$$

$$v = \sum_{n=1}^{n=\infty} v_n = \sum_{n=1}^{n=\infty} \hat{V}_n \cos (nt + \phi_n) \tag{1007}$$

be respectively the current flowing through the impedance and the fall of potential across its terminals, the instantaneous rate of energy consumption in the impedance is

$$p = iv \tag{1008}$$

$$= \left\{ \sum_{n=1}^{n=\infty} \hat{I}_n \cos (n\omega t + \theta_n) \right\} \left\{ \sum_{n=1}^{n=\infty} \hat{V}_n \cos (n\omega t + \phi_n) \right\} \tag{1009}$$

$$= \sum \frac{\hat{I}_m \hat{V}_n}{2} \cos (m\omega t + \theta_m) \cos (n\omega t + \phi_n), \tag{1010}$$

the summation including every possible combination of +ve integral values of m and n.

$$p = \sum \frac{\hat{I}_m \hat{V}_n}{2} \cos \left\{ (m - n)\omega t + (\theta_m - \phi_n) \right. \tag{1011}$$

$$\left. + \sum \frac{\hat{I}_m \hat{V}_n}{2} \cos \left\{ (m + n)\omega t + (\theta_m + \phi_n) \right\} \right. . \tag{1012}$$

The first of these summations can be put in the form

$$\sum \frac{\hat{I}_m \hat{V}_n}{2} \cos \left\{ (m - n)\omega t + (\theta_m - \phi_n) \right\} = \sum \frac{\mathbf{I}_m \cdot \mathbf{V}_n}{2}. \tag{1013}$$

$$= \frac{\mathbf{I} \cdot \mathbf{V}}{2}. \tag{1014}$$

Comparing Eq. (1011) with the general expression for the product of two scalar products given in Par. 32, it is seen that the second summation must be the instantaneous value of $\mathbf{W} \cdot \boldsymbol{\nu}$, $\mathbf{W}$ being a vector of magnitude $\frac{\hat{I}\hat{V}}{2}$, making with $\boldsymbol{\nu}$ an angle equal to the sum of the angles made by $\mathbf{I}$ and $\mathbf{V}$ with $\boldsymbol{\nu}$, *i.e.*,

$$p = \frac{\mathbf{I} \cdot \mathbf{V}}{2} + \mathbf{W} \cdot \boldsymbol{\nu}, \tag{1015}$$

where, by reference to Eq. (1011), $\mathbf{W}$ is seen to be a periodic vector containing no constant term, no fundamental term, and, in general, all the harmonics from the second onwards.

In the more usual practical case in which $\mathbf{I}$ and $\mathbf{V}$ contain odd harmonics only, $\mathbf{W}$ will only contain harmonics of order 2, 4, 6, 8, etc., etc.

*Mean Power.*—For the mean value of the power over a period,

$$P = \frac{1}{T}\int_0^T p\, dt = \frac{1}{T}\int_0^T \frac{(\mathbf{I} \cdot \mathbf{V})}{2} dt + \frac{1}{T}\int_0^T (\mathbf{W} \cdot \boldsymbol{\nu}) dt \tag{1016}$$

and, since $\mathbf{W}$ has no constant component,

$$P = \frac{1}{T}\int_0^T \frac{(\mathbf{I} \cdot \mathbf{V})}{2} dt = \frac{1}{2T}\int_0^T (\mathbf{I} \cdot \mathbf{V}) dt. \tag{1017}$$

The expression for the mean power is, therefore, essentially the same as that which applies to pure sine-wave currents and potentials. For the mean value of $\mathbf{I} \cdot \mathbf{V}$

$$\mathbf{I} \cdot \mathbf{V} = \sum_{n=1}^{n=\infty} \mathbf{I}_n \cdot \mathbf{V}_n + \sum \mathbf{I}_m \cdot \mathbf{V}_n, \tag{1018}$$

the second summation including all possible combinations of +ve integral values of m and n except m = n. For a typical term of this second summation,

$$\mathbf{I}_m \cdot \mathbf{V}_n = \hat{\mathbf{I}}_m \hat{\mathbf{V}}_n \cos{(m - n)\omega t + (\theta_m - \phi_n)}. \tag{1019}$$

Since (m − n) is a whole number, the mean value of $\mathbf{I}_m \cdot \mathbf{V}_n$ over a period is zero. Of the first summation, however, every term is a constant, for

$$\mathbf{I}_n \cdot \mathbf{V}_n = \hat{\mathbf{I}}_n \hat{\mathbf{V}}_n \cos{(\theta_n - \phi_n)}. \tag{1020}$$

Thus, the mean value of the first summation of Eq. (1018) is $\sum_{n=1}^{n=\infty} \mathbf{I}_n \cdot \mathbf{V}_n$, so that

$$P = \tfrac{1}{2} \sum_{n=1}^{n=\infty} \mathbf{I}_n \cdot \mathbf{V}_n. \tag{1021}$$

This is in accordance with the conclusion reached in Par. 68, namely, that *in considering alternating currents and potentials of general wave form each harmonic can be taken separately and the total result obtained by the summation of the separate results for the individual harmonics.*

For the relation between the above expression and the value of the impedance

$$\mathbf{V}_n = \mathbf{I}_n z_n. \tag{1022}$$

Therefore,

$$P = \tfrac{1}{2} \sum_{n=1}^{n=\infty} (z_n \mathbf{I}_n \cdot \mathbf{I}_n) \tag{1023}$$

$$= \tfrac{1}{2} \sum_{n=1}^{n=\infty} (R_n + jX_n)\mathbf{I}_n \cdot \mathbf{I}_n \tag{1024}$$

$$= \tfrac{1}{2} \sum_{n=1}^{n=\infty} R_n(\mathbf{I}_n \cdot \mathbf{I}_n) \tag{1025}$$

$$= \tfrac{1}{2} \sum_{n=1}^{n=\infty} R_n \hat{\mathbf{I}}_n^2. \tag{1026}$$

If $R_n$ is const., and of magnitude R, then, putting

$$I_n = \text{root-mean-square value of } i_n \tag{1027}$$

$$= \frac{\hat{I}_n}{\sqrt{2}} \tag{1028}$$

$$P = R \sum_{n=1}^{n=\infty} I_n^2. \tag{1029}$$

Expressing the same result in terms of V,

$$P = \frac{1}{2} \sum_{n=1}^{n=\infty} \frac{\mathbf{V}_n}{\mathbf{Z}_n} \cdot \mathbf{V}_n \tag{1030}$$

$$= \frac{1}{2} \sum_{n=1}^{n=\infty} \frac{R_n - jX_n}{R_n^2 + X_n^2} \mathbf{V}_n \cdot \mathbf{V}_n \tag{1031}$$

$$= \frac{1}{2} \sum_{n=1}^{n=\infty} \frac{R_n}{R_n^2 + X_n^2} \hat{V}_n^2 \tag{1032}$$

$$= \sum_{n=1}^{n=\infty} \frac{R_n}{R_n^2 + X_n^2} V_n^2, \tag{1033}$$

where $V_n$ is the root-mean-square value of $v_n$, the nth harmonic of v.

It should be noted that the quantities involved in the above two expressions for the mean power in terms of the impedance are $\sum_{n=1}^{n=\infty} I_n^2$ and $\sum_{n=1}^{n=\infty} V_n^2$ respectively, and that these are quite independent of the phase relationships between the harmonics and the fundamental. Thus, as far as the power consumed in the impedance $z$ is concerned, the two currents $\sum_{n=1}^{n=\infty} \hat{I}_n \cos(n\omega t + \theta_n)$ and $\sum_{n=1}^{n=\infty} \hat{I}_n \cos n\omega t$ are identical.

**71. The Root-mean-square Value of Alternating Currents and Potentials of General Wave Form.**—In the majority of the practical applications of alternating currents and potentials, the important quantity is the mean value of the square of the current or the potential over a period.

Putting
$$i = \mathbf{I} \cdot \boldsymbol{\nu}, \tag{1034}$$
then
$$i^2 = (\mathbf{I} \cdot \boldsymbol{\nu})(\mathbf{I} \cdot \boldsymbol{\nu}) \tag{1035}$$
and, as shown in Par. 32,
$$(\mathbf{I} \cdot \boldsymbol{\nu})(\mathbf{I} \cdot \boldsymbol{\nu}) = \frac{\mathbf{I} \cdot \mathbf{I}}{2} + \mathbf{W} \cdot \boldsymbol{\nu}. \tag{1036}$$

It can be shown exactly as in the preceding paragraph that **W** is a periodic vector containing no constant component, *i.e.*,

its mean value over a period is zero, also that the mean value of the remainder of the expression is

$$\frac{1}{2}\sum_{n=1}^{n=\infty} \mathrm{I}_n \cdot \mathrm{I}_n = \frac{1}{2}\sum_{n=1}^{n=\infty} \hat{\mathrm{I}}_n^2 \tag{1037}$$

$$= \sum_{n=1}^{n=\infty} \mathrm{I}_n^2 \tag{1038}$$

where $\mathrm{I}_n$ is the root-mean-square value of $i_n$, the nth harmonic.

Thus, putting I for the square root of the mean value of $i^2$ over a period,

$$\mathrm{I}^2 = \sum_{n=1}^{n=\infty} \mathrm{I}_n^2, \tag{1039}$$

*.e.,*

$$\mathrm{I} = \sqrt[2]{\sum_{n=1}^{n=\infty} \mathrm{I}_n^2}. \tag{1040}$$

In a similar manner it can be shown that V, the square root of he mean value of v over a period, is

$$\mathrm{V} = \sqrt[2]{\sum_{n=1}^{n=\infty} \mathrm{V}_n^2}. \tag{1041}$$

The quantities I and V are known as the root-mean-square alues of the alternating current and the potential respectively. *is clear that they depend on the relative magnitudes of the har- onics, and not on the phase relationships between them.* These e quantities which are measured by ordinary alternating- rrent instruments, and these quantities are used to designate e value of the alternating currents and the potentials concerned.

Referring to the preceding paragraph, it is seen that the mean ower consumed in the impedance $z$ can be expressed

$$\mathrm{P} = \mathrm{RI}^2, \tag{1042}$$

being the root-mean-square value of the current. This is, course, identical with the expression that would apply to the se of a continuous current of magnitude I flowing through the me total resistance.

**72. Power Factor.**—Referring to Fig. 70, an alternating-current ameter connected in series with the impedance $z$ would show a ading

$$\mathrm{I} = \left\{\sum_{n=1}^{n=\infty} \mathrm{I}_n^2\right\}^{\frac{1}{2}}. \tag{1043}$$

An alternating-current voltmeter connected across its terminals would show a reading

$$V = \left\{ \sum_{n=1}^{n=\infty} \hat{V}_n^2 \right\}^{\frac{1}{2}}. \tag{1044}$$

A wattmeter connected in the usual way with its current coil in series with $z$ and its potential winding across the terminals of $z$ would indicate the mean value of the product of i and v, *i.e.*, as shown in Par. 70, the average value of **I** · **V**.

If i and v were of pure sine-wave form, then

$$P = \frac{\mathbf{I} \cdot \mathbf{V}}{2} \tag{1045}$$

$$= \frac{\hat{I}\hat{V}}{2} \cos \theta \tag{1046}$$

$$= IV \cos \theta, \tag{1047}$$

I and V being the root-mean-square values of i and v.

In this connection, $\cos \theta$ is termed the "power factor" of the load $z$. It is, of course, the constant angle between the vectors **I** and **V**, *i.e.*, the angular phase difference between i and v.

In dealing with currents and potentials of general wave form, these simple relationships no longer hold. The angle between **I** and **V** is not a constant, but is itself a periodic function of time. For convenience, however, the same notation is retained in the general case, *i.e.*,

$$\frac{P}{IV} = \text{power factor} \tag{1048}$$

both for sine-wave forms and for the general wave form. It must be realized, however, that, whereas in the case of sine-wave forms, the power factor depends only on the load $\left(\text{since } \tan \theta = \frac{X}{R}\right)$, in the general case it depends not only on the load but also on the wave form of the e.m.f. which supplies current to it. In fact,

if R is const., $$P = RI^2, \tag{1049}$$

so that $$\text{Power factor} = \frac{RI^2}{IV} \tag{1050}$$

$$= R \frac{I}{V} \tag{1051}$$

$$= R \left\{ \frac{\Sigma I_n^2}{\Sigma V_n^2} \right\}^{\frac{1}{2}}. \tag{1052}$$

ɜut
$$\frac{I_n}{V_n} = \frac{1}{\{R^2 + X_n^2\}^{\frac{1}{2}}}, \tag{1053}$$

*e.*,
$$I_n^2 = \frac{V_n^2}{R^2 + X_n^2}. \tag{1054}$$

'herefore,
$$\text{Power factor} = R \left\{ \frac{\sum_{n=1}^{n=\infty} \frac{V_n^2}{R^2 + X_n^2}}{\sum_{n=1}^{n=\infty} V_n^2} \right\}^{\frac{1}{2}}, \tag{1055}$$

› that the power factor depends not only on R and X but also ı the relative magnitudes of the harmonics of v. It should be ›ted, however, that it depends only on the relative magnitudes the harmonics, and not on the relative phases of the harmonics. is, moreover, independent of the absolute magnitude of v, ovided the harmonics are always of the same relative magnitude.

**73. The Identity and the Equivalence of Alternating Currents General Wave Form.**—Two currents will be considered

$$i = \sum_{n=1}^{n=\infty} \hat{I} \cos (n\omega t + \theta) \tag{1056}$$

$$i' = \sum_{n=1}^{n=\infty} \hat{I}' \cos (n\omega t + \theta') \tag{1057}$$

›resented respectively by the vectors **I** and **I′** where

$$\mathbf{I} = \sum_{n=1}^{n=\infty} \mathbf{I}_n \tag{1058}$$

$$\mathbf{I}' = \sum_{n=1}^{n=\infty} \mathbf{I}'_n. \tag{1059}$$

en if

$$\mathbf{I} = \mathbf{I}' \tag{1060}$$

every instant, the currents i and i′ are equal in every respect, in magnitude, wave form, and phase.

ɹet $\mathbf{I}_t$ be the vector **I** at the instant t. Then, if for all values

$$\mathbf{I}_t = \mathbf{I}'_{t+rT} \tag{1061}$$

ɜre r is some positive number less than one, then i and i′ are al in magnitude and in wave form, but there is a phase differ- ɜ of r of a period between them.

If $$\mathbf{I} \neq \mathbf{I}' \tag{1062}$$

but $$\hat{I}_n = \hat{I}_n' \tag{1063}$$

for all values of n, then, from Par. 71,

$$I = I', \tag{1064}$$

*i.e.*, the currents i and i′ are not of the same wave form, but their effective or root-mean-square values are the same.

Similarly, for the inequalities of alternating currents of general wave form, if

$$\mathbf{I} = k\mathbf{I}' \tag{1065}$$

at every instant, then the currents i and i′ are similar in wave form and phase, but i is k times i′ in magnitude.

Again, if $$\mathbf{I}_t = k\mathbf{I}'_{t+rT}, \tag{1066}$$

the currents i and i′ are similar in wave form, but i is k times i′ in magnitude, and r of a period ahead of it in phase.

If $$\mathbf{I} \neq k\mathbf{I}' \tag{1067}$$

but $$\hat{I}_n = k\hat{I}_n' \tag{1068}$$

for all values of n, then $$I = kI' \tag{1069}$$

*i.e.*, the currents i and i′ are dissimilar in wave form, but the effective or root-mean-square value of i is k times that of i′.

Restating these conclusions in scalar form, we have, as the conditions for the complete equality of the currents i and i′, are

$$\hat{I}_n = \hat{I}_n' \tag{1070}$$

$$\theta_n = \theta_n' \tag{1071}$$

both of which are implied in the single vector equation above.

Putting $$\mathbf{I}_t' \cdot \boldsymbol{\nu} = \sum_{n=1}^{n=\infty} \hat{I}_n' \cos(n\omega t + \theta_n') \tag{1072}$$

then $$\mathbf{I}'_{t+rT} \cdot \boldsymbol{\nu} = \sum_{n=1}^{n=\infty} \hat{I}_n' \cos\{n\omega(t + rT) + \theta_n'\} \tag{1073}$$

and since $$\omega = \frac{2\pi}{T} \tag{1074}$$

$$\mathbf{I}'_{t+rT} = \sum_{n=1}^{n=\infty} \hat{I}_n' \cos(n\omega t + n\omega rT + \theta_n') \tag{1075}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n' \cos(n\omega t + 2\pi nr + \theta_n') \tag{1076}$$

$$= \sum_{n=1}^{n=\infty} \epsilon^{j2\pi nr}\, \mathbf{I}_n' \cdot \boldsymbol{\nu} \tag{1077}$$

$$= \sum_{n=1}^{n=\infty} \epsilon^{jn\phi}\, \mathbf{I}_n' \cdot \boldsymbol{\nu} \tag{1078}$$

where $$\phi = \frac{2\pi}{r}. \tag{1079}$$

Thus, the condition for equality of magnitude and of wave form with a phase difference of r of a period is

$$\sum_{n=1}^{n=\infty} \mathbf{I}_n = \sum_{n=1}^{n=\infty} \epsilon^{jn\phi}\, \mathbf{I}_n' \tag{1080}$$

which, in scalar terms, becomes

$$\hat{I}_n = \hat{I}_n' \tag{1081}$$

$$\theta_n = \theta_n' + n\phi, \tag{1082}$$

*i.e.*,

$$\frac{\theta_n - \theta_n'}{n} = \text{const.} = \phi. \tag{1083}$$

On the other hand, the condition for effective or root-mean-square equality without equality of wave form is

$$\hat{I}_n = \hat{I}_n' \tag{1084}$$

$$\frac{\theta_n - \theta_n'}{n} = \text{not const.} \tag{1085}$$

It should be noted that only with currents of similar wave form is it possible to assign a value for angular phase difference, and even in these cases the angle between the two vectors is constant and equal to this angle. If the current represented by the vector **I** is described as having an angular phase difference $\phi$ with respect to another current represented by the vector **I**$'$, this means that the angle between the vectors representing the nth harmonics of these currents is $n\phi$.

Further, note should be taken of the distinction between

$$\mathbf{I} = \sum_{n=1}^{n=\infty} \mathbf{I}_n = \sum_{n=1}^{n=\infty} \epsilon^{jn\phi}\, \mathbf{I}_n' \tag{1086}$$

and

$$\mathbf{I} = \sum_{n=1}^{n=\infty} \mathbf{I}_n = \sum_{n=1}^{n=\infty} \epsilon^{j\psi}\, \mathbf{I}_n' \tag{1087}$$

$$= \epsilon^{j\psi} \sum \mathbf{I}_n'. \tag{1088}$$

In the first case there is an angular phase difference $\phi$ between the currents i and i$'$. In the second case

$$i = \sum_{n=1}^{n=\infty} \mathbf{I}_n \cdot \boldsymbol{\nu} \tag{1089}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n \cos(n\omega t + \theta_n) \tag{1090}$$

$$i' = \sum_{n=1}^{n=\infty} \mathbf{I}_n' \cdot \boldsymbol{\nu} \tag{1091}$$

$$= \sum_{n=1}^{n=\infty} \hat{I}_n' \cos(n\omega t + \theta_n + \psi), \tag{1092}$$

which means that i and i$'$ are equal in effective or root-mean-square value but are of different wave form. Also the angle between the vectors **I** and **I**$'$ is constant and equal to $\psi$.

## EXAMPLES

**1.** An e.m.f. represented by

$$e = 100 \cos 100\pi t + 10 \cos (300\pi t + 30°) + 10 \cos (500\pi t + 60°) + 5 \cos (700\pi t\ 90°) \text{ volts}$$
$$= (\mathbf{E}_1 + \mathbf{E}_3 + \mathbf{E}_5 + \mathbf{E}_7) \cdot \mathbf{v}$$

acts in a circuit consisting of an inductance of 10 henries having a resistance of 5 ohms, in series with a pure capacity of 0.0405 microfarads.

(*a*) Express the current flowing in the circuit
  (1) In vector form in terms of $\mathbf{E}_1$, $\mathbf{E}_3$, $\mathbf{E}_5$, and $\mathbf{E}$.
  (2) In scalar form.

(*b*) Compare the ratios of the amplitudes of the harmonics in the e.m.f. to that of the fundamental, with the corresponding ratios for the current.

**2.** Find the mean power consumed in the above circuit under the conditions specified.

**3.** (*a*) A current represented by

$$i = 100 \cos 100\pi t + 20 \cos (300\pi t + 15°)$$
$$20 \cos (500\pi t + 60°) + 10 \cos (700\pi t + 10°) + \cos 900\pi t \text{ amperes}$$

is flowing in a pure resistance of 25 ohms. Find the mean power consumed.

(*b*) What is the mean power consumed if the resistance is not independent of frequency, its magnitude at a frequency f being given by

$$R_f = 25 + \frac{f}{50} + \frac{f^2}{2{,}000}?$$

**4.** An e.m.f. represented by

$$e = 100 \cos 100\pi t + 15 \cos (300\pi t + 30°)$$
$$+ 20 \cos (500\pi t + 60°) + 10 \cos (700\pi t + 10°) \text{ volts}$$

supplies current to a circuit consisting of a resistance of 100 ohms in parallel with a pure inductance of 0.159 henries. Calculate:

(*a*) The r.m.s. value of the e.m.f.
(*b*) The r.m.s. value of the current.
(*c*) The mean power consumed in the load.
(*d*) The power factor.

## ANSWERS TO EXAMPLES

**1.** (*a*) (1) $\mathbf{I} = \frac{\epsilon^{j90°}}{75{,}550}\mathbf{E}_1 + \frac{\epsilon^{j90°}}{16{,}700}\mathbf{E}_3 + \frac{1}{5}\mathbf{E}_5 + \frac{\epsilon^{-j90°}}{10{,}800}\mathbf{E}_7.$

(2) $i = 1.325 \times 10^{-3} \cos (100\pi t + 90°)$
$+.6 \times 10^{-3} \cos (300\pi t + 120°)$
$+2 \cos (500\pi t + 60°)$
$+4.625 \times 10^{-4} \cos 700\pi t.$

(*b*)
$$\frac{\hat{E}_3}{\hat{E}_1} = .1\ ;\ \frac{\hat{I}_3}{\hat{I}_1} = .4525$$
$$\frac{\hat{E}_5}{\hat{E}_1} = .1\ ;\ \frac{\hat{I}_5}{\hat{I}_1} = 1510.0$$
$$\frac{\hat{E}_7}{\hat{E}_1} = .05;\ \frac{\hat{I}_7}{\hat{I}_1} = .349.$$

**2.** Putting

$$P = P_1 + P_2 + P_3 + P_4$$

$P_1$, $P_3$ $P_7$ are negligible, being of the order of microwatts.

$$P = P_5 = 10 \text{ watts.}$$

**3.** (*a*) 136.26 kilowatts.
(*b*) 161.08 kilowatts.

**4.** (*a*) 73.3 volts. (*b*) 1.594 amperes.
(*c*) 53.63 watts. (*d*) .46.

# CHAPTER VIII

## SYMMETRICAL POLYPHASE SYSTEMS[1]

**74. The Definition of a Symmetrical m-phase System E.M.Fs.**—A series of m e.m.fs., equal in magnitude and in wa form, between successive members of which there exists a tin phase difference of $\frac{1}{m^{th}}$ of a period, is said to constitute a syn metrical m-phase system.

Thus if

$$e = \sum_{n=1}^{n=\infty} \hat{E}_n \cos(n\omega t + \theta_n) = \sum_{n=1}^{n=\infty} \mathbf{E}_n \cdot \boldsymbol{\nu} = \mathbf{E} \cdot \boldsymbol{\nu}, \quad (109$$

then

$$\mathbf{E}_{(1)} = \sum_{n=1}^{n=\infty} \mathbf{E}_n \quad (109$$

$$\mathbf{E}_{(2)} = \sum_{n=1}^{n=\infty} \epsilon^{jn\phi}\mathbf{E}_n \quad (109$$

$$\mathbf{E}_{(3)} = \sum_{n=1}^{n=\infty} \epsilon^{j2n\phi}\mathbf{E}_n \quad (109$$

$$\mathbf{E}_{(4)} = \sum_{h=1}^{n=\infty} \epsilon^{j3n\phi}\mathbf{E}_n \quad (109$$

$$\mathbf{E}_{(m)} = \sum_{n=1}^{n=\infty} \epsilon^{j(m-1)n\phi}\mathbf{E}_n \quad (109$$

where

$$\phi = \frac{2\pi}{m} \quad (109$$

are the successive members of a symmetrical m-phase syster

*Note.*—In the above equations (1094 to 1098) the subscripts indicating t number of the phase have been enclosed in brackets in order to distingui them from similar subscripts indicating harmonic order. In the followi paragraphs, in which pure sine waves only will be considered, these bracke will be omitted for the sake of simplicity.

[1] Bibliography, Nos. 9, 20, and 17.

**75. The Symmetrical Three-phase System of Sine-wave E.M.Fs.**—A full discussion of the general m-phase system defined as above would be of theoretical rather than of practical interest. Certain points of the general theory will be considered later in this chapter, but for the present, in view of the practical object of this book, it will be preferable to confine the discussion to the most important of the polyphase systems, namely, the three-phase system of which the constituent e.m.fs. can be considered to be approximately of sine-wave form.

Putting $$e = \hat{E} \cos \omega t = \mathbf{E} \cdot \mathbf{v}, \tag{1100}$$

then $$\mathbf{E}_1 = \mathbf{E} \tag{1101}$$

$$\mathbf{E}_2 = \epsilon^{j2\frac{\pi}{3}}\mathbf{E} = \epsilon^{j120°}\mathbf{E} \tag{1102}$$

$$\mathbf{E}_3 = \epsilon^{j4\frac{\pi}{3}}\mathbf{E} = \epsilon^{j240°}\mathbf{E} \tag{1103}$$

will be the vectors which represent the e.m.fs. of a symmetrical three-phase system.

In scalar form $$e_1 = \mathbf{E}_1 \cdot \mathbf{v} = \hat{E} \cos \omega t \tag{1104}$$

$$e_2 = \mathbf{E}_2 \cdot \mathbf{v} = \hat{E} \cos (\omega t + 120°) \tag{1105}$$

$$e_3 = \mathbf{E}_3 \cdot \mathbf{v} = \hat{E} \cos (\omega t + 240°). \tag{1106}$$

**76. Graphical Representation of a Three-phase System.**—The three vectors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ can, obviously, be represented by three lines of equal magnitude inclined to each other at an angle of 120°, as shown in Fig. 71.

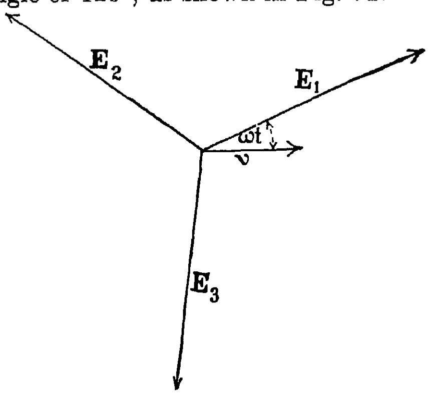


FIG. 71.

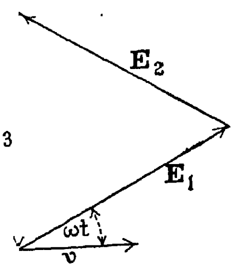


FIG. 72.

**77. The Sum of the E.M.Fs. of the Three-phase System.**—Since the vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ can be represented, as shown in Fig. 72, by the three sides on an equilateral triangle,

$$\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = 0. \tag{1107}$$

Alternatively, $$\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = (1 + \epsilon^{j120°} + \epsilon^{j240°})\mathbf{E} \quad (1108)$$
$$= \{(1 + \cos 120° + \cos 240°) + j(\sin 120° + \sin 240°)\}\mathbf{E} \quad (1109)$$
$$= \left\{\left(1 - \frac{1}{2} - \frac{1}{2}\right) + j\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right\}\mathbf{E} \quad (1110)$$
$$= 0, \quad (1111)$$
*i.e.*, $$(1 + \epsilon^{j120°} + \epsilon^{j240°}) = 0 \quad (1112)$$

**78. The Interconnection of Three-phase E.M.Fs.**—In practice, the e.m.fs. of a three-phase system are derived ultimately

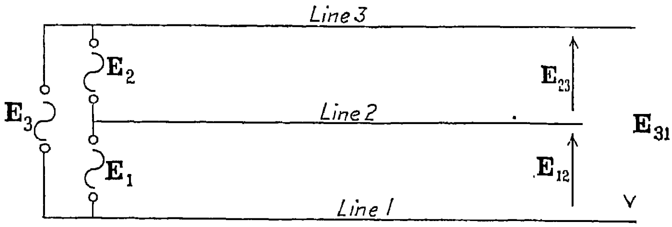


FIG. 73.

from the armature windings of a three-phase generator, though they may, of course, pass through various transformers or other apparatus before being utilized as a source of power.

For reasons of economy in conductors polyphase systems are usually interconnected. For this purpose two main alternatives

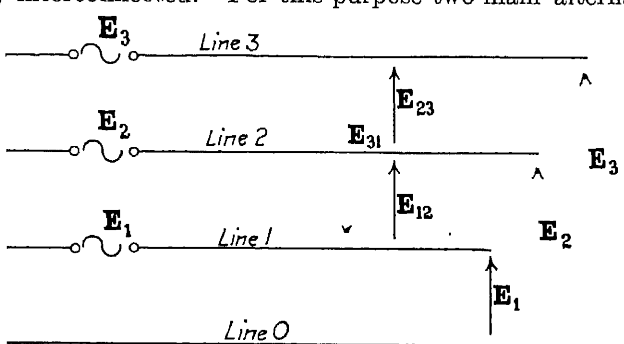


FIG. 74.

are available, the choice between them depending on the practical conditions and requirements in any given case.

The first is known as the "mesh" or "ring" (or in the case of three-phase, "delta") connection. The second is known as the "star" connection. These connections are illustrated in Figs.

73 and 74 respectively. The figures also show the symbols which will be used in the discussion of the two systems. In addition, the following symbols will be used.

$E_P$ = r.m.s. value of the phase e.m.f.

$E_L$ = r.m.s. value of the line e.m.f.

**79. Three-phase Operators.**—In the discussion of three-phase systems certain operators will be of frequent occurrence. For

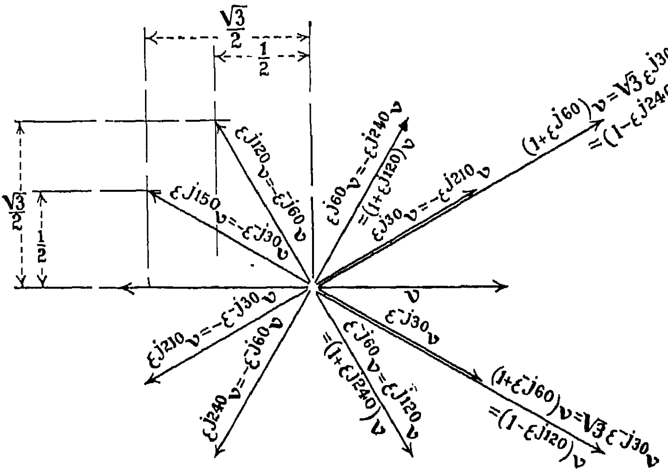


FIG. 75.

convenience of reference, these are given below with their equivalent forms (see also Fig. 75).

$$\epsilon^{j120^\circ} = \epsilon^{-j240^\circ} = -\epsilon^{-j60^\circ} = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \tag{1113}$$

$$\epsilon^{j240^\circ} = \epsilon^{-j120^\circ} = -\epsilon^{j60^\circ} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \tag{1114}$$

$$\epsilon^{-j30^\circ} = -\epsilon^{j150^\circ} = \frac{\sqrt{3}}{2} - j\frac{1}{2} \tag{1115}$$

$$\epsilon^{j30^\circ} = -\epsilon^{j210^\circ} = \frac{\sqrt{3}}{2} + j\frac{1}{2} \tag{1116}$$

$$1 - \epsilon^{j120^\circ} = 1 + \epsilon^{-j60^\circ} = \sqrt{3}\epsilon^{-j30^\circ} = \sqrt{3}\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) \tag{1117}$$

$$1 + \epsilon^{j120^\circ} = -\epsilon^{j240^\circ} = \epsilon^{j60} = \frac{1}{2} + j\frac{\sqrt{3}}{2} \tag{1118}$$

$$1 + \epsilon^{j60} = \sqrt{3}\epsilon^{j30^\circ} = \frac{3}{2} + j\frac{\sqrt{3}}{2} \tag{1119}$$

$$1 - \epsilon^{j60} = \epsilon^{-j60^\circ} = \frac{1}{2} - j\frac{\sqrt{3}}{2}. \tag{1120}$$

**80. Star-connected Three-phase System.**—One important difference between the two types of interconnection mentioned above is immediately obvious from a consideration of Fig. 74. *With delta connection, the e.m.fs. operating between the supply lines are the terminal potential differences of the phases. With star connection, on the other hand, the line e.m.fs. are the differences between the terminal p.d.s. of successive phases, i.e.,*

$$\mathbf{E}_{12} = \mathbf{E}_1 - \mathbf{E}_2 \tag{1121}$$

$$\mathbf{E}_{23} = \mathbf{E}_2 - \mathbf{E}_3 \tag{1122}$$

$$\mathbf{E}_{31} = \mathbf{E}_3 - \mathbf{E}_1. \tag{1123}$$

It follows from this that

$$\mathbf{E}_{12} + \mathbf{E}_{23} + \mathbf{E}_{31} = 0 \tag{1124}$$

quite independently of

$$\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = 0. \tag{1125}$$

The line e.m.fs. can also be expressed

$$\mathbf{E}_{12} = (1 - \epsilon^{j120})\mathbf{E}_1 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_1 \tag{1126}$$

$$\mathbf{E}_{23} = (1 - \epsilon^{j120})\mathbf{E}_2 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_2 \tag{1127}$$

$$\mathbf{E}_{31} = (1 - \epsilon^{j120})\mathbf{E}_3 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_3 \tag{1128}$$

Therefore,

$$E_L = \sqrt{3}\,E_P. \tag{1129}$$

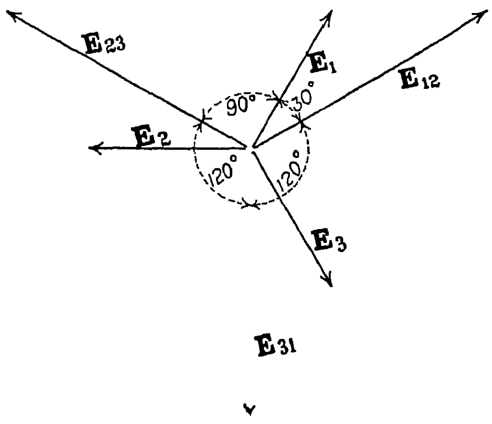


FIG. 76.

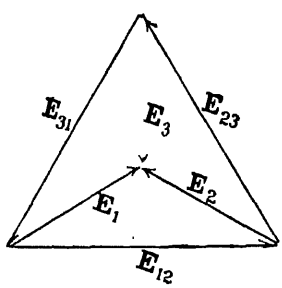


FIG. 77.

The above combinations are illustrated graphically in Figs. 76 and 77.

**81. Delta-connected Three-phase System.**—In this case

$$E_{12} = E_1 \tag{1130}$$

$$E_{23} = E_2 \tag{1131}$$

$$E_{31} = E_3. \tag{1132}$$

It should be noted that the phase windings constitute a closed circuit for the phase e.m.fs. in series. Since, however

$$E_1 + E_2 + E_3 = 0, \tag{1133}$$

there is no resultant e.m.f. acting around this closed circuit.

**82. Three-phase Line Currents.**—The symbols which will be used in the discussion of the line currents are shown in Figs.

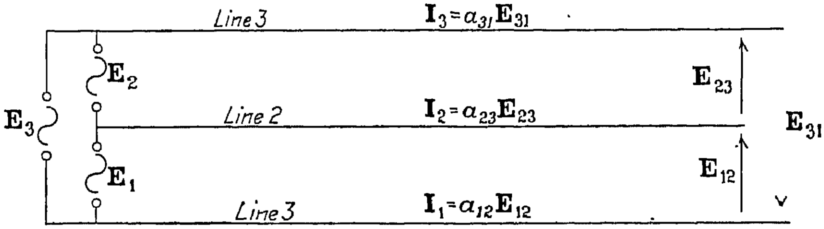


FIG. 78.

78 and 79. In addition, $I_1$, $I_2$, $I_3$ will be used for the r.m.s. values of the line currents and $I_L$ for the r.m.s. line current in balanced systems.

Efficient voltage regulation will be assumed, *i.e.*, the line and phase e.m.fs. will be assumed to remain constant for all values

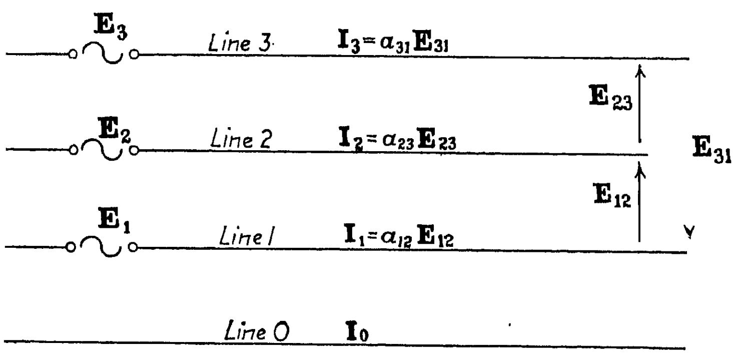


FIG. 79.

of the line currents. This will greatly simplify the analysis without materially detracting from its value.

1. *Star-connected Supply System.*—This is illustrated in Fig. 79. There may or may not be a fourth or neutral line to the neutral point of the supply. If there is, by applying Kirchhoff's first law to the neutral point,

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_0 = 0 \tag{1134}$$

or

$$\mathbf{I}_0 = -(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3). \tag{1135}$$

It will be convenient for many purposes to express the line currents in terms of the corresponding line or phase e.m.f. and an operator of the type $r\epsilon^{j\theta}$ or $(a + jb)$.
For this purpose the following symbols will be used:

$$\mathbf{I}_1 = a_1\mathbf{E}_1 = a_{12}\mathbf{E}_{12} \tag{1136}$$

$$\mathbf{I}_2 = a_2\mathbf{E}_2 = a_{23}\mathbf{E}_{23} \tag{1137}$$

$$\mathbf{I}_3 = a_3\mathbf{E}_3 = a_{31}\mathbf{E}_{31} \tag{1138}$$

also

$$\mathbf{I}_0 = a_0\mathbf{E}_1 \tag{1139}$$

where

$$a_1 = \epsilon^{j\psi_1}\mathrm{a}_1 = (\mathrm{g}_1 + j\mathrm{s}_1) \tag{1140}$$

$$a_{12} = \epsilon^{j\psi_{12}}\mathrm{a}_{12} = (\mathrm{g}_{12} + j\mathrm{s}_{12}) \tag{1141}$$

and similarly for the other a terms.

The above operators are not, of course, independent for, since

$$\mathbf{E}_{12} = \sqrt{3}\epsilon^{-j30}\mathbf{E}_1, \tag{1142}$$

$$\sqrt{3}\epsilon^{-j30}a_{12} = a_1 \text{ etc., etc.} \tag{1143}$$

There is, moreover, a relation between the operators of either set, for, from Eq. (1134),

$$a_0\mathbf{E}_1 + a_1\mathbf{E}_1 + a_2\mathbf{E}_2 + a_3\mathbf{E}_3 = 0. \tag{1144}$$

Therefore,

$$(a_0 + a_1 + \epsilon^{j120}a_2 + \epsilon^{j240}a_3)\mathbf{E}_1 = 0, \tag{1145}$$

*i.e.*,

$$(a_0 + a_1 + \epsilon^{j120}a_2 + \epsilon^{j240}a_3) = 0. \tag{1146}$$

It follows from this that, if there is no neutral line,

$$a_1 + \epsilon^{j120}a_2 + \epsilon^{j240}a_3 = 0 \tag{1147}$$

or, alternatively,

$$a_{12} + \epsilon^{j120}a_{23} + \epsilon^{j240}a_{31} = 0. \tag{1148}$$

2. *Delta-connected Supply.*—In this case the line currents $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ are the differences between the currents flowing in the successive phases of the supply. It follows from this (*cf.* Eqs. (1121) to (1124)) that

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 0. \tag{1149}$$

There is, therefore, the relation between the line operators

$$a_{12} + \epsilon^{j120} a_{23} + \epsilon^{j240} a_{31} = 0. \qquad (1150)$$

**83. Three-phase Loads.**[1]—The loads to which three-phase systems of e.m.f. are applied can be interconnected in either

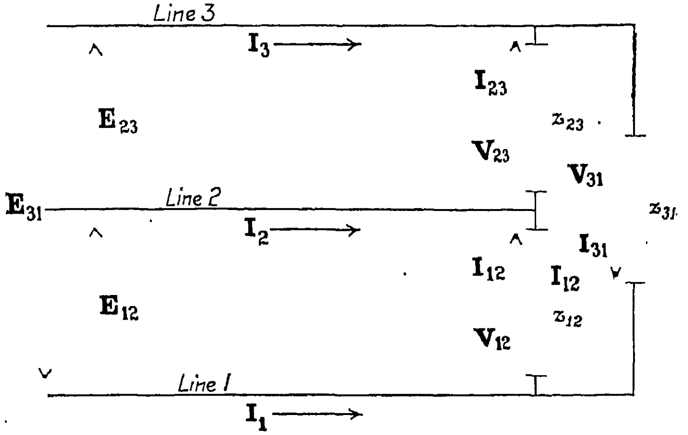


FIG. 80.

of the two ways already described. The two types of connection and the symbols which will be used in their discussion are illus-

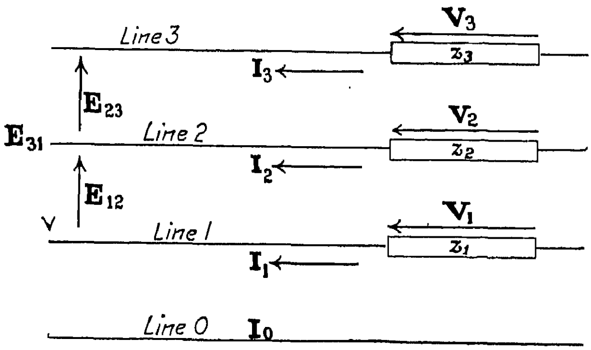


FIG. 81.

trated in Figs. 80 and 81. The loads $z_1$, $z_{12}$, etc. are general impedances of the type

$$z_1 = z_1\epsilon^{j\theta_1} = R_1 + jX_1 \qquad (1151)$$

$$z_{12} = z_{12}\epsilon^{j\theta_{12}} = R_{12} + jX_{12} \qquad (1152)$$

etc., etc.

[1] Bibliography Nos. 9, 17.

It will be convenient in some cases to express these loads as admittances, in which case the following symbols will be employed:

$$\frac{1}{z_1} = y_1 = \mathrm{y}_1\epsilon^{-j\theta_1} = \mathrm{G}_1 + j\mathrm{S}_1 \tag{1153}$$

$$\frac{1}{z_{12}} = y_{12} = \mathrm{y}_{12}\epsilon^{-j\theta_{12}} = \mathrm{G}_{12} + j\mathrm{S}_{12} \tag{1154}$$

etc., etc.

It should be noted that the signs attributed to $\mathbf{V}_1$, $\mathbf{V}_{12}$, etc., etc., the potential differences across the loads, are such that

$$\mathbf{E}_{12} + \mathbf{V}_{12} = 0 \tag{1155}$$

$$\mathbf{E}_{12} + \mathbf{V}_1 - \mathbf{V}_2 = 0 \tag{1156}$$

etc., etc.

In the discussion of the load equations it is immaterial whether the supply system be star- or delta-connected, except in so far as this concerns the possibility or otherwise of a fourth line. The line e.m.fs. being $\mathbf{E}_{12}$, $\mathbf{E}_{23}$, $\mathbf{E}_{31}$, then, if desired, use can be made of the symbols $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, on the understanding that

$$\mathbf{E}_1 = \frac{1}{\sqrt{3}}\epsilon^{+j30}\mathbf{E}_{12} \tag{1157}$$

etc., etc.;

*i.e.*, if the supply system is a star-connected one, then $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are the phase terminal potential differences. If not, then they are the phase terminal potential differences of the equivalent star-connected system.

The first question in the discussion of three-phase loads is the relation between the line currents and the line e.m.fs., *i.e.*, the expression of the line operators $a_1$, $a_{12}$, etc. in terms of the loads $z_1$, $z_{12}$, etc. This will be the subject of the following paragraphs.

**84. Three-line Star-connected Unbalanced Load.**—This is illustrated in Fig. 81, the neutral line being disregarded.

By the application of Kirchhoff's second law,

$$\mathbf{E}_{12} = -\mathbf{V}_1 + \mathbf{V}_2 \tag{1158}$$

$$\mathbf{E}_{23} = -\mathbf{V}_2 + \mathbf{V}_3 \tag{1159}$$

$$\mathbf{E}_{31} = -\mathbf{V}_3 + \mathbf{V}_1. \tag{1160}$$

The potential differences of the system are, therefore, as shown in Fig. 82, the potential of the neutral point of the load with respect to the real or assumed neutral point of the supply being represented by the vector $\mathbf{V}_0$.

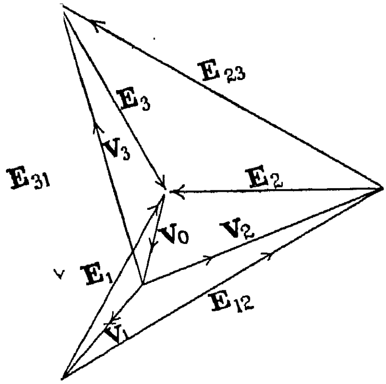


FIG. 82.

For this vector $\mathbf{V}_0$,

$$\mathbf{V}_1 + \mathbf{E}_1 + \mathbf{V}_0 = 0 \tag{1161}$$

$$\mathbf{V}_2 + \mathbf{E}_2 + \mathbf{V}_0 = 0 \tag{1162}$$

$$\mathbf{V}_3 + \mathbf{E}_3 + \mathbf{V}_0 = 0, \tag{1163}$$

whence, by addition,

$$3(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3) + 3(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3) + 3\mathbf{V}_0 = 0. \tag{1164}$$

Therefore, $$\mathbf{V}_0 = -(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3). \tag{1165}$$

For the line currents in terms of the line e.m.fs.

since $$\mathbf{V}_1 = -z_1\mathbf{I}_1 \tag{1166}$$

$$\mathbf{V}_2 = -z_2\mathbf{I}_2 \tag{1167}$$

$$\mathbf{V}_3 = -z_3\mathbf{I}_3 \tag{1168}$$

therefore $$z_1\mathbf{I}_1 - z_2\mathbf{I}_2 = \mathbf{E}_{12} \tag{1169}$$

$$z_2\mathbf{I}_2 - z_3\mathbf{I}_3 = \mathbf{E}_{23} \tag{1170}$$

$$z_3\mathbf{I}_3 - z_1\mathbf{I}_1 = \mathbf{E}_{31}. \tag{1171}$$

These three equations are not independent, however, for the third can be derived from the other two. For their solution, therefore, a fourth equation is required. This can be obtained by the application of Kirchhoff's first law to the neutral point, which gives

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 0. \tag{1172}$$

The solution of the above equations by the ordinary algebraic methods will be found to be

$$\mathbf{I}_1 = \frac{z_3\mathbf{E}_{12} - z_2\mathbf{E}_{31}}{z_c} \tag{1173}$$

$$\mathbf{I}_2 = \frac{z_1\mathbf{E}_{23} - z_3\mathbf{E}_{12}}{z_c} \tag{1174}$$

$$\mathbf{I}_3 = \frac{z_2\mathbf{E}_{31} - z_1\mathbf{E}_{23}}{z_c}, \tag{1175}$$

where

$$z_c = z_1z_2 + z_2z_3 + z_3z_1. \tag{1176}$$

For the expression of $a_{12}$, $a_{23}$, $a_{31}$ in terms of the loads $z_1$, $z_2$, $z_3$, however, it is necessary to know $\mathbf{I}_1$ in terms of $\mathbf{E}_{12}$, $\mathbf{I}_2$ in terms of $\mathbf{E}_{23}$, and $\mathbf{I}_3$ in terms of $\mathbf{E}_{31}$. For this purpose

$$\mathbf{E}_{31} = \epsilon^{j240}\mathbf{E}_{12} = -\epsilon^{j60}\mathbf{E}_{12} \tag{1177}$$

$$\mathbf{E}_{12} = \epsilon^{-j120}\mathbf{E}_{23} = -\epsilon^{j60}\mathbf{E}_{23} \tag{1178}$$

$$\mathbf{E}_{23} = \epsilon^{-j120}\mathbf{E}_{31} = -\epsilon^{j60}\mathbf{E}_{31}. \tag{1179}$$

Therefore,

$$\mathbf{I}_1 = \frac{z_3 + \epsilon^{j60}z_2}{z_c}\mathbf{E}_{12} \tag{1180}$$

$$\mathbf{I}_2 = \frac{z_1 + \epsilon^{j60}z_3}{z_c}\mathbf{E}_{23} \tag{1181}$$

$$\mathbf{I}_3 = \frac{z_2 + \epsilon^{j60}z_1}{z_c}\mathbf{E}_{31}, \tag{1182}$$

*i.e.*,

$$a_{12} = \frac{z_3 + \epsilon^{j60}z_2}{z_c} \tag{1183}$$

$$a_{23} = \frac{z_1 + \epsilon^{j60}z_3}{z_c} \tag{1184}$$

$$a_{31} = \frac{z_2 + \epsilon^{j60}z_1}{z_c}. \tag{1185}$$

If it is desired to express the line operators in terms of the resistance and the reactance components of the loads, then

$$\begin{aligned} z_c &= \{(R_1 + jX_1)(R_2 + jX_2) + (R_2 + jX_2)(R_3 + jX_3) \\ &\qquad + (R_3 + jX_3)(R_1 + jX_1)\} \qquad (1186) \\ &= R_c + jX_c, \end{aligned}$$

where

$$R_c = (R_1R_2 + R_2R_3 + R_3R_1) - (X_1X_2 + X_2X_3 + X_3X_1) \tag{1187}$$

and

$$X_c = (R_1X_2 + R_2X_3 + R_3X_1) + (X_1R_2 + X_2R_3 + X_3R_1). \tag{1188}$$

atively, $$z_c = z_c\epsilon^{j\psi_c} \tag{1189}$$

$$z_c^2 = R_c^2 + X_c^2 \tag{1190}$$

$$\psi_c = \tan^{-1}\frac{X_c}{R_c}. \tag{1191}$$

rly, $$z_3 + \epsilon^{j60}z_2 = z_3 + \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)z_2 \tag{1192}$$

$$_3 + \frac{1}{2}R_2 - X_2\frac{\sqrt{3}}{2}\Big) + j\left(X_3 + \frac{1}{2}X_2 + R_2\frac{\sqrt{3}}{2}\right) \tag{1193}$$
$$= k_{12}\epsilon^{j\phi12}$$

$$^2 + z_2^2 + (R_2R_3 + X_2X_3) - \sqrt{3}(R_3X_2 - R_2X_3) \tag{1194}$$

$$\tan\phi_{12} = \frac{2X_3 + X_2 + \sqrt{3}R_2}{2R_3 + R_2 - \sqrt{3}X_2}. \tag{1195}$$

$$a_{12} = \left\{\frac{k_{12}}{z_c}\right\}\epsilon^{j(\phi_{12} - \psi_c)} \tag{1196}$$

ilarly, for $a_{23}$ and $a_{31}$.
xpressions involved are seen to be simple in character somewhat lengthy in form. In practice, graphical provide an easier way of obtaining the solution in any se. For this purpose it will be found more convenient with admittances rather than impedances. Dividing 30), (1181), and (1182) by $z_1z_2z_3$ and putting

$$\frac{1}{z_1} = y_1 \tag{1197}$$

$$\frac{1}{z_2} = y_2 \tag{1198}$$

$$\frac{1}{z_3} = y_3 \tag{1199}$$

$$y_c = y_1 + y_2 + y_3, \tag{1200}$$

essions become

$$a_{12} = \frac{y_1y_2 + \epsilon^{j60}y_1y_3}{y_c} \tag{1201}$$

$$a_{23} = \frac{y_2y_3 + \epsilon^{j60}y_2y_1}{y_c} \tag{1202}$$

$$a_{31} = \frac{y_3y_1 + \epsilon^{j60}y_3y_2}{y_c}. \tag{1203}$$

raphical determination, based on the methods described 9, is illustrated in Fig. 83 for the values

$$y_1 = 1.165\epsilon^{j8°40'}$$
$$y_2 = 1.5\epsilon^{j24°}$$
$$y_3 = 1.6\epsilon^{j37°}.$$

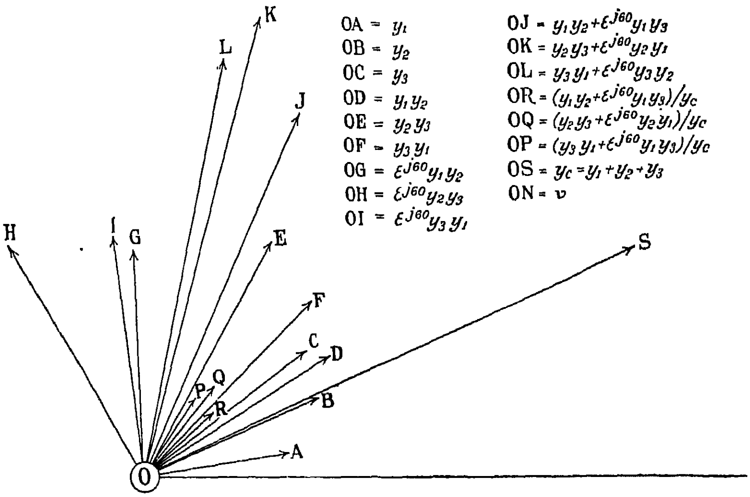


FIG. 83.

The corresponding values of $y_c$ and of the line operators were found by the graphical determination to be

$$y_c = 4.16\epsilon^{j24°55'}$$
$$a_{12} = .745\epsilon^{j41°35'}$$
$$a_{23} = .88\epsilon^{j51°}$$
$$a_{31} = .72\epsilon^{j56°5'}.$$

*Special Cases.*—1. Non-reactive Loads.—If

$$X_1 = X_2 = X_3 = 0, \qquad (1204)$$

then
$$R_c = R_1R_2 + R_2R_3 + R_3R_1 \qquad (1205)$$

and
$$X_c = 0, \qquad (1206)$$

*i e.*,
$$z_c = R_c \qquad (1207)$$
$$\psi_c = 0. \qquad (1208)$$

Further,
$$z_3 + \epsilon^{j60} z_2 = R_3 + \frac{1}{2}R_2 + j\frac{\sqrt{3}}{2}R_2 \qquad (1209)$$

so that
$$k_{12}^2 = R_2^2 + R_3^2 + R_2R_3. \qquad (1210)$$

Therefore,
$$a_{12} = \frac{\sqrt{R_2^2 + R_3^2 + R_2R_3}}{R_1R_2 + R_2R_3 + R_3R_1} \qquad (1211)$$

and
$$\psi_{12} = \tan^{-1}\frac{\sqrt{3}R_2}{2R_3 + R_2}. \qquad (1212)$$

2. Reactive Loads.—If

$$R_1 = R_2 = R_3 = 0, \qquad (1213)$$

then $$R_c = -(X_1X_2 + X_2X_3 + X_3X_1) \text{ or } X_c = 0, \tag{1214}$$
so that $$z_c = -R_c \tag{1215}$$
$$\psi_c = \pi. \tag{1216}$$

Further $$z_3 + \epsilon^{j60} z_2 = -\frac{\sqrt{3}}{2X_2 + j(X_3 + \frac{1}{2}X_2)}, \tag{1217}$$
so that $$k_{12}^2 = X_3^2 + X_2^2 + X_3X_2 \tag{1218}$$
$$\phi_{12} = \tan^{-1} - \frac{\sqrt{3}X_2}{(2X_3 + X_2)}. \tag{1219}$$

The question of the position of the neutral point of the load in the potential-difference diagram is of practical interest.

If Eqs. (1180), (1181) and (1182) are divided by $z_1z_2z_3$, then substituting admittances for impedances,

$$I_1 = \frac{y_1y_2 + \epsilon^{j60}y_1y_3}{y_c} E_{12} \tag{1220}$$
$$I_2 = \frac{y_2y_3 + \epsilon^{j60}y_2y_1}{y_c} E_{23} \tag{1221}$$
$$I_3 = \frac{y_3y_1 + \epsilon^{j60}y_3y_2}{y_c} E_{31} \tag{1222}$$

where $$y_c = y_1 + y_2 + y_3. \tag{1223}$$
Now, since $$V_1 = -z_1I_1 \tag{1224}$$
$$= -\frac{I_1}{y_1} \tag{1225}$$
etc., etc.,

then, from Eqs. (1220) to (1222),

$$V_1 = -\frac{(y_2 + \epsilon^{j60}y_3)}{y_c} E_{12} \tag{1226}$$
$$V_2 = -\frac{(y_3 + \epsilon^{j60}y_1)}{y_c} E_{23} \tag{1227}$$
$$V_3 = -\frac{(y_1 + \epsilon^{j60}y_2)}{y_c} E_{31} \tag{1228}$$

and since $$-V_0 = \tfrac{1}{3}(V_1 + V_2 + V_3). \tag{1229}$$

Therefore, $$3y_cV_0 = (y_2 + \epsilon^{j60}y_3)E_{12} + (y_3 + \epsilon^{j60}y_1)E_{23} + (y_1 + \epsilon^{j60}y_2)E_{31} \tag{1230}$$
$$= \sqrt{3}\{\epsilon^{-j30}(y_2 + \epsilon^{j60}y_3) + \epsilon^{j90}(y_3 + \epsilon^{j60}y_1) + \epsilon^{j210}(y_1 + \epsilon^{j60}y_2)\}E_1 \tag{1231}$$
$$= \sqrt{3}\{(\epsilon^{j150} + \epsilon^{j210})y_1 + (\epsilon^{j-30} + \epsilon^{j270})y_2 + (\epsilon^{j30} + \epsilon^{j90})y_3\}E_1. \tag{1232}$$

By drawing these operators it can easily be shown that

$$\epsilon^{j150} + \epsilon^{j210} = -\sqrt{3} \tag{1233}$$

$$\epsilon^{-j30} + \epsilon^{j270} = -\sqrt{3}\epsilon^{j120} \tag{1234}$$

$$\epsilon^{j30} + \epsilon^{j90} = -\sqrt{3}\epsilon^{j240} \tag{1235}$$

Therefore,
$$3y_c\mathbf{V}_0 = -3(y_1 + \epsilon^{j120}y_2 + \epsilon^{j240}y_3)\mathbf{E}_1 \tag{1236}$$

or
$$-y_c\mathbf{V}_0 = y_1\mathbf{E}_1 + y_2\mathbf{E}_2 + y_3\mathbf{E}_3. \tag{1237}$$

*In other words, the vector* $-y_c\mathbf{V}_0$ is *the resultant of the vectors* $y_1\mathbf{E}_1$, $y_2\mathbf{E}_2$, $y_3\mathbf{E}_3$. The graphical determination of $\mathbf{V}_0$, *i.e.*, of the position of the neutral point, is thus a simple matter. It is illustrated in Fig. 84 for the special values already considered.

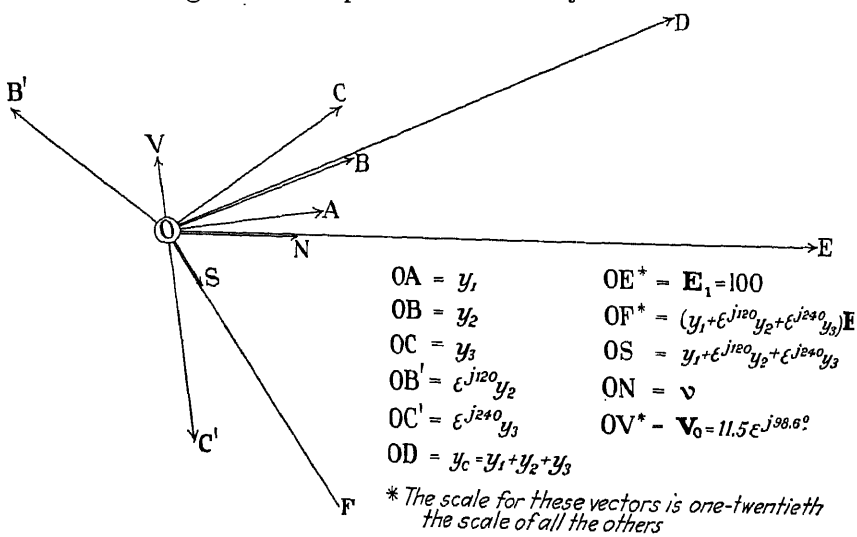


FIG. 84.

**85. Three-line Star-connected Balanced Load.**—Putting

$$z_1 = z_2 = z_3 = z_s \tag{1238}$$

$$y_1 = y_2 = y_3 = y_s \tag{1239}$$

then, from Eqs. (1180) to (1182),

$$\mathbf{I}_1 = (1 + \epsilon^{j60})\frac{\mathbf{E}_{12}}{3z_s} \tag{1240}$$

$$\mathbf{I}_2 = (1 + \epsilon^{j60})\frac{\mathbf{E}_{23}}{3z_s} \tag{1241}$$

$$\mathbf{I}_3 = (1 + \epsilon^{j60})\frac{\mathbf{E}_{31}}{3z_s} \tag{1242}$$

and, since
$$\frac{(1 + \epsilon^{j60})}{3} = \frac{\epsilon^{j30}}{\sqrt{3}}, \tag{1243}$$

the above equations can be written

$$\mathbf{I}_1 = \frac{\mathbf{E}_1}{z_s} \tag{1244}$$

$$\mathbf{I}_2 = \frac{\mathbf{E}_2}{z_s} \tag{1245}$$

$$\mathbf{I}_3 = \frac{\mathbf{E}_3}{z_s}, \tag{1246}$$

*i.e.,*

$$a_1 = a_2 = a_3 = y_s. \tag{1247}$$

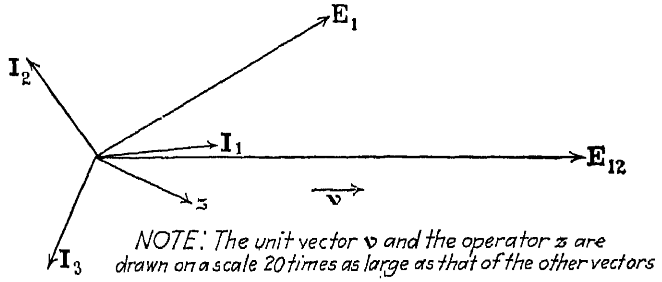


FIG. 85.

Therefore, the vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, are equal to $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ respectively, and $\mathbf{V}_0 = 0$. The case is illustrated in Fig. 85 for the values

$$\hat{\mathbf{E}}_{12} = 200$$
$$z = 2.24\epsilon^{-j26°}.$$

**86. Four-line Star-connected Unbalanced Load.**—This is illustrated in Fig. 81. The current and potential relationships are simple in this case, for

$$\mathbf{E}_1 = \mathbf{V}_1 \tag{1248}$$

$$\mathbf{E}_2 = \mathbf{V}_2 \tag{1249}$$

$$\mathbf{E}_3 = \mathbf{V}_3 \tag{1250}$$

and

$$\mathbf{V}_0 = 0. \tag{1251}$$

Therefore,

$$a_1 = y_1 \tag{1252}$$

$$a_2 = y_2 \tag{1253}$$

$$a_3 = y_3 \tag{1254}$$

For the current in the neutral line

$$\mathbf{I}_0 = -(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3), \tag{1255}$$

*i.e.,*

$$\mathbf{I}_0 = -(y_1E_1 + y_2E_2 + y_3E_3). \tag{1256}$$

Comparing this with Eq. (1237) it is seen that the current in the neutral line is that which would be produced in a circuit consisting of the three admittances $y_1$, $y_2$, $y_3$ connected in parallel, by an e.m.f. equal to the potential of the neutral point in the three-line case considered in Par. 84.

**87. Delta-connected Unbalanced Load.**—This is illustrated in Fig. 80. It will be seen that

$$\mathbf{I}_1 = \mathbf{I}_{12} - \mathbf{I}_{31} \tag{1257}$$

$$\mathbf{I}_2 = \mathbf{I}_{23} - \mathbf{I}_{12} \tag{1258}$$

$$\mathbf{I}_3 = \mathbf{I}_{31} - \mathbf{I}_{23}. \tag{1259}$$

Also,

$$\mathbf{I}_{12} = y_{12}\mathbf{E}_{12} \tag{1260}$$

$$\mathbf{I}_{23} = y_{23}\mathbf{E}_{23} \tag{1261}$$

$$\mathbf{I}_{31} = y_{31}\mathbf{E}_{31}. \tag{1262}$$

Therefore,

$$\mathbf{I}_1 = y_{12}\mathbf{E}_{12} - y_{31}\mathbf{E}_{31} = (y_{12} + \epsilon^{j60}y_{31})\mathbf{E}_{12} \tag{1263}$$

$$\mathbf{I}_2 = y_{23}\mathbf{E}_{23} - y_{12}\mathbf{E}_{12} = (y_{23} + \epsilon^{j60}y_{12})\mathbf{E}_{23} \tag{1264}$$

$$\mathbf{I}_3 = y_{31}\mathbf{E}_{31} - y_{23}\mathbf{E}_{23} = (y_{31} + \epsilon^{j60}y_{23})\mathbf{E}_{31}, \tag{1265}$$

*i.e.*,

$$a_{12} = y_{12} + \epsilon^{j60}y_{31} \tag{1266}$$

$$= \left(G_{12} + \frac{1}{2}G_{31} - \frac{\sqrt{3}}{2}S_{31}\right) + j\left(S_{12} + \frac{1}{2}S_{31} + \frac{\sqrt{3}}{2}G_{31}\right) \tag{1267}$$

and similarly for $a_{23}$ and $a_{31}$.

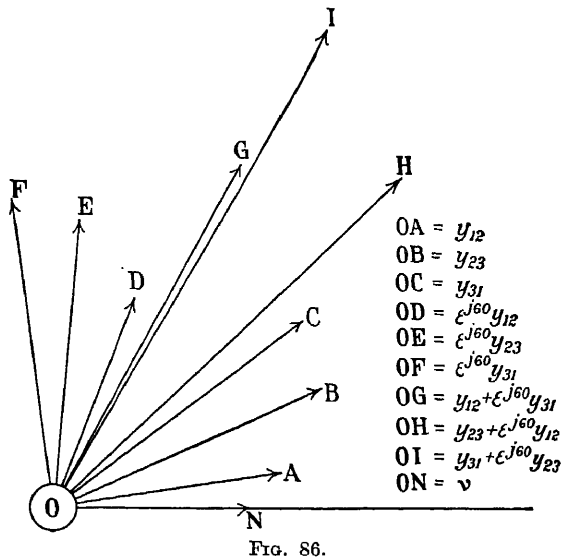


Fig. 86.

The corresponding vector diagram is given in Fig. 86 for the numerical values already considered in Par. 84,

*i.e.*,

$$z_{12} = \frac{1}{1.165}\epsilon^{-j8°40'}$$

$$z_{23} = \frac{1}{1.5}\epsilon^{-j24°}$$

$$z_{31} = \frac{1}{1.6}\epsilon^{-j37°}$$

which give (*cf*. Fig. 83)

$a_{12} = 2.05\epsilon^{j61°30'}$ $\quad a_{23} = 2.5\epsilon^{j43°30'}$ $\quad a_{31} = 2.9\epsilon^{j60°}$.

**88. Delta-connected Balanced Load.**—If the load is balanced,

$$y_{12} = y_{23} = y_{31} = y_m. \tag{1268}$$

Therefore,

$$\mathbf{I}_1 = (1 + \epsilon^{j60})y_m\mathbf{E}_{12} = \sqrt{3}\epsilon^{j30}y_m\mathbf{E}_{12} \tag{1269}$$

$$\mathbf{I}_2 = (1 + \epsilon^{j60})y_m\mathbf{E}_{23} = \sqrt{3}\epsilon^{j30}y_m\mathbf{E}_{23} \tag{1270}$$

$$\mathbf{I}_3 = (1 + \epsilon^{j60})y_m\mathbf{E}_{31} = \sqrt{3}\epsilon^{j30}y_m\mathbf{E}_{31}. \tag{1271}$$

The three-line currents are, therefore, equal in magnitude and

$$\mathrm{I_L} = \sqrt{3}\mathrm{y_m E_L} = \sqrt{3}\,\frac{\mathrm{E_L}}{\mathrm{z_m}}. \tag{1272}$$

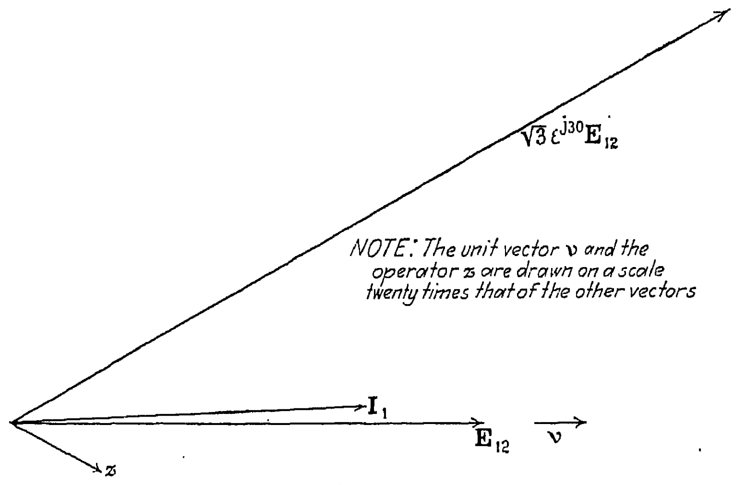


FIG. 87.

The case is illustrated in Fig. 87 for the values

$$\hat{\mathbf{E}}_{12} = 200$$

$$z = 2.24\epsilon^{-j26°}$$

(*cf*. Fig. 85).

**89. The Equivalence of Star- and Delta-connected Loads.**—1. *Star, Three-line, and Delta-connected Loads.*—Two loads, one three-line star-connected and the other delta-connected, can be said to be equivalent if the line-current vectors $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ are the same in each case.

Comparing Eqs. (1180) to (1182) with Eqs. (1263) to (1265), it is seen that the loads $z_1$, $z_2$, $z_3$ connected in lines 1, 2, and 3 will be exactly equivalent to loads $y_{12}$, $y_{23}$, $y_{31}$ connected between lines 1 and 2, 2 and 3, 3 and 1, if

$$\frac{z_3}{z_c} = y_{12} = \frac{1}{z_{12}} \tag{1273}$$

$$\frac{z_1}{z_c} = y_{23} = \frac{1}{z_{23}} \tag{1274}$$

$$\frac{z_2}{z_c} = y_{31} = \frac{1}{z_{31}}, \tag{1275}$$

*i.e.*,

$$y_{12} = \frac{y_1 y_2}{y_c} \tag{1276}$$

$$y_{23} = \frac{y_2 y_3}{y_c} \tag{1277}$$

$$y_{31} = \frac{y_3 y_1}{y_c} \tag{1278}$$

Alternatively, if it is required to express $y_1$, $y_2$, and $y_3$ in terms of $y_{12}$, $y_{23}$, and $y_{31}$, from the above three equations

$$\frac{y_{31} y_{12}}{y_{23}} = \frac{y_1^2}{y_c} \tag{1279}$$

$$y_{31} + y_{12} = \frac{y_1(y_2 + y_3)}{y_c}. \tag{1280}$$

Therefore,

$$y_{12} + y_{31} + \frac{y_{12} y_{31}}{y_{23}} = \frac{y_1(y_1 + y_2 + y_3)}{y_c} \tag{1281}$$

$$= y_1 \tag{1282}$$

since

$$y_c = y_1 + y_2 + y_3. \tag{1283}$$

Therefore,

$$y_1 = y_{12} + y_{31} + \frac{y_{12} y_{31}}{y_{23}} \tag{1284}$$

$$y_2 = y_{23} + y_{12} + \frac{y_{23} y_{12}}{y_{31}} \tag{1285}$$

$$y_3 = y_{31} + y_{23} + \frac{y_{31} y_{23}}{y_{12}}. \tag{1286}$$

These equivalent loads are illustrated in Fig. 88.

If the loads are simple in character, *e.g.*, non-reactive or wholly reactive, the calculation of the equivalent load is a simple matter. For instance, if

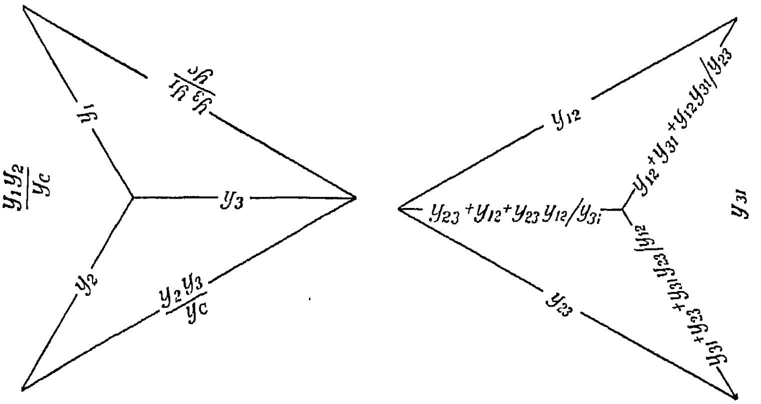


FIG. 88.

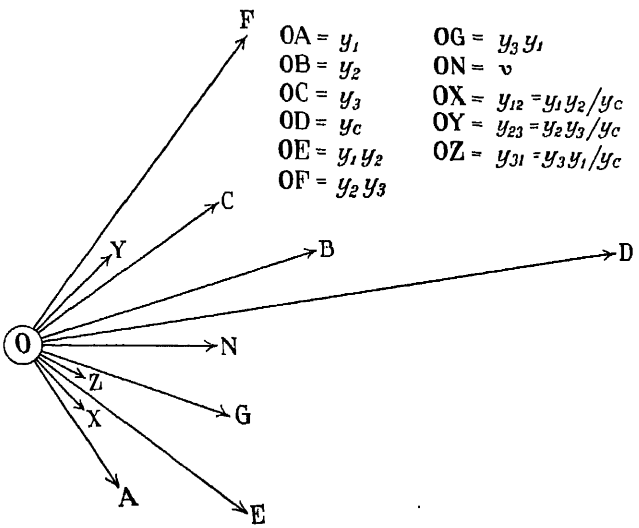


FIG. 89.

$$y_1 = .025 \text{ reciprocal ohms}$$
$$y_2 = .100 \text{ reciprocal ohms}$$
$$y_3 = .175 \text{ reciprocal ohms}$$

then $y_c = .200$ reciprocal ohms

$$y_{12} = \frac{.0025}{.2} = .0125 \text{ reciprocal ohms}$$

$$y_{23} = \frac{.0175}{.2} = .0825 \text{ reciprocal ohms}$$

$$y_{31} = \frac{.004375}{.2} = .021875 \text{ reciprocal ohms.}$$

In a more general case a graphical method of solution is preferable, and is illustrated in Fig. 89 for the values

$$y_1 = .9\epsilon^{-j55°36'}$$
$$y_2 = 1.59\epsilon^{j18°36'}$$
$$y_3 = 1.25\epsilon^{j37°},$$

the equivalent delta-connected loads being

$$y_{12} = .47\epsilon^{-j46°15'}$$
$$y_{23} = .65\epsilon^{j46°}$$
$$y_{31} = .37\epsilon^{-j27°}.$$

2. *Equivalence of Balanced Star- and Delta-connected Loads.*—
If
$$y_1 = y_2 = y_3 = y_s \tag{1287}$$
and
$$y_{12} = y_{23} = y_{31} = y_m, \tag{1288}$$
then the equivalence condition becomes
$$3y_m = y_s, \tag{1289}$$
*i.e.*,
$$3R_s = R_m \tag{1290}$$
$$3X_s = X_m \tag{1291}$$
or
$$3z_s = z_m \tag{1292}$$
$$\theta_s = \theta_m. \tag{1293}$$

**90. The Determination of Loads in Terms of Line Operators.**—Given the line operators $a_{12}$, $a_{23}$, $a_{31}$ completely, *i.e.*, in magnitude and angle, is it possible to determine the star- or delta-connected load which will give these values of the line operators?

It is immaterial whether the load be determined as either star- or delta-connected, since the equivalence conditions give a means of transformation from the one to the other. Assuming, therefore, that the load is a delta-connected one, the equations (see Par. 87) are

$$a_{12} = y_{12} + \epsilon^{j60}y_{31} \tag{1294}$$
$$a_{23} = y_{23} + \epsilon^{j60}y_{12} \tag{1295}$$
$$a_{31} = y_{31} + \epsilon^{j60}y_{23}. \tag{1296}$$

It would appear that here are three equations for the determination of the three unknown quantities $y_{12}$, $y_{23}$, $y_{31}$. The equations are not, however, independent, for

$$a_{12} + \epsilon^{j120}a_{23} + \epsilon^{j240}a_{31} = 0. \tag{1297}$$

Multiplying the first equation by $\epsilon^{j60}$ and subtracting the second from the result,

$$\epsilon^{j120}y_{31} - y_{23} = \epsilon^{j60}a_{12} - a_{23}. \tag{1298}$$

Multiplying this by $\epsilon^{j240}$

$$y_{31} + \epsilon^{j60}y_{23} = \epsilon^{j300}a_{12} - \epsilon^{j240}a_{23} \tag{1299}$$

$$= -\epsilon^{j120}a_{12} - \epsilon^{j240}a_{23} \tag{1300}$$

$$= a_{31}$$

from Eq. (1297).

Thus the third equation can be derived from the other two. The data are, therefore, insufficient for the specification of the loads, since an infinite number of values of $y_{12}$, $y_{23}$, and $y_{31}$ can be found which will satisfy the first two equations, and these will necessarily satisfy the remaining equation.

*It may therefore be said thât in the general case of a three-line unbalanced load a knowledge of the magnitudes and the phases of the three-line currents is not sufficient to determine the loads.*

**91. Power in Three-phase Systems.**[1]—Consider a supply system such as that illustrated in Fig. 90, in which the phase potential differences $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are maintained constant in ampli-

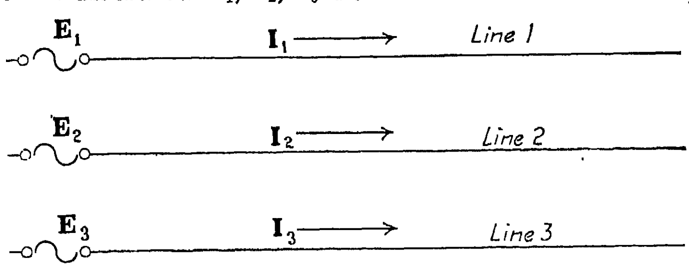


FIG. 90.

tude. As shown in Par. 32, the instantaneous rate at which energy is being supplied to the lines by the terminal potential differences $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ can be expressed in the form

$$p = \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3) + (\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3) \cdot \boldsymbol{\nu}, \tag{1301}$$

where $\mathbf{W}_1$ is a vector of magnitude $\frac{\hat{I}_1\hat{E}_1}{2}$, which makes with $\boldsymbol{\nu}$

[1] Bibliography, No. 14.

an angle equal to the sum of the angles made by $\mathbf{I}_1$ and $\mathbf{E}_1$ with $\boldsymbol{\nu}$, and similarly for $\mathbf{W}_2$ and $\mathbf{W}_3$.

The line potential differences produced by this system will be, as indicated in the figure, $\mathbf{E}_{12}$, $\mathbf{E}_{23}$, $\mathbf{E}_{31}$ between lines 1 and 2, 2 and 3, and 3 and 1 respectively, where

$$\mathbf{E}_{12} = \mathbf{E}_1 - \mathbf{E}_2 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_1 \tag{1302}$$

$$\mathbf{E}_{23} = \mathbf{E}_2 - \mathbf{E}_3 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_2 \tag{1303}$$

$$\mathbf{E}_{31} = \mathbf{E}_3 - \mathbf{E}_1 = \sqrt{3}\epsilon^{-j30}\mathbf{E}_3. \tag{1304}$$

Now, as shown in Par. 83, given these constant line potential differences, it is immaterial whether the actual supply system is star- or delta-connected. In either case Eq. (1301) in conjunction with Eqs. (1302), (1303), (1304) will accurately represent the instantaneous power being supplied to the lines. Equation (1301) will, therefore, be taken as the standard form for the discussion of three-phase power.

As in the simpler case of a single alternating e.m.f. supplying current to a circuit, the expression for the instantaneous power is seen to consist of two parts, one being the constant term

$$\tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3) \tag{1305}$$

and the other the periodic term

$$(\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3), \tag{1306}$$

which itself consists of the sum of three periodic terms derived one from each of the three phases. The mean value of this latter term over a period is zero, so that the mean rate of energy supply to the lines is given by

$$P = \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3). \tag{1307}$$

In practice this is the important factor. Before considering it in detail, however, it will be well to discuss the periodic term a little more fully, since it possesses some features of special interest.

**92. The Periodic Term in Three-phase Power.**—Since

$$\mathbf{I}_1 = a_1\mathbf{E}_1 = a_1\epsilon^{j\psi_1}\mathbf{E}_1 \tag{1308}$$

and similarly for $\mathbf{I}_2$ and $\mathbf{I}_3$, then

$$\mathbf{W}_1 \cdot \boldsymbol{\nu} = \frac{\hat{I}_1\hat{E}_2}{2}\cos(2\omega t + \psi_1) \tag{1309}$$

$$= \frac{a_1\hat{E}_1^2}{2}\cos(2\omega t + \psi_1) \tag{1310}$$

$$= a_1E_p^2\cos(2\omega t + \psi_1) \tag{1311}$$

and similarly

$$\mathbf{W}_2 \cdot \boldsymbol{\nu} = a_2E_p^2\cos(2\omega t + \psi_2 + 240°) \tag{1312}$$

$$\mathbf{W}_3 \cdot \boldsymbol{\nu} = a_3E_p^2\cos(2\omega t + \psi_3 + 120°). \tag{1313}$$

If $\mathbf{W}_0$ be a vector of magnitude $E_p^2$, velocity $2\omega$, and initial phase zero, *i.e.*,

$$\mathbf{W}_0 \cdot \nu = E_p^2 \cos 2\omega t, \tag{1314}$$

then from Eqs. (1311), (1312), and (1313)

$$\mathbf{W}_1 = a_1\mathbf{W}_0 \tag{1315}$$

$$\mathbf{W}_2 = \epsilon^{j240}a_2\mathbf{W}_0 \tag{1316}$$

$$\mathbf{W}_3 = \epsilon^{j120}a_3\mathbf{W}_0, \tag{1317}$$

so that $$\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = (a_1 + \epsilon^{j240}a_2 + \epsilon^{j120}a_3)\mathbf{W}_0 \tag{1318}$$

$$= \sqrt{3}\epsilon^{-j30}(a_{12} + \epsilon^{j290}a_{23} + \epsilon^{j120}a_{31})\mathbf{W}_0. \tag{1319}$$

*Special Cases.*—1. Unbalanced Three-line Star-connected Non-inductive Load.—Putting

$$z_1 = R_1 \tag{1320}$$

$$z_2 = R_2 \tag{1321}$$

$$z_3 = R_3, \tag{1322}$$

from Eqs. (1183), (1184) and (1185)

$$a_{12} = \frac{(R_3 + \epsilon^{j60}R_2)}{R_c} \tag{1323}$$

$$a_{23} = \frac{(R_1 + \epsilon^{j60}R_3)}{R_c} \tag{1324}$$

$$a_{31} = \frac{(R_2 + \epsilon^{j60}R_1)}{R_c} \tag{1325}$$

where $$R_c = (R_1R_2 + R_2R_3 + R_3R_1). \tag{1326}$$

Therefore, $(a_{12} + \epsilon^{j240}a_{23} + \epsilon^{j120}a_{31}) =$

$$\frac{(R_3 + \epsilon^{j240}R_1 + \epsilon^{j120}R_2) + \epsilon^{j60}(R_2 + \epsilon^{j240}R_3 + \epsilon^{j120}R_1)}{R_c} \tag{1327}$$

$$= \epsilon^{j180}(1 + \epsilon^{j60})\frac{(R_1 + \epsilon^{-j120}R_2 + \epsilon^{-j120}R_3)}{R_c} \tag{1328}$$

$$= -\sqrt{3}\epsilon^{j30}\frac{(R_1 + \epsilon^{j240}R_2 + \epsilon^{j120}R_3)}{R_c}. \tag{1329}$$

Also, $$R_1 + \epsilon^{j240}R_2 + \epsilon^{j120}R_3 = R_1 - \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)R_2 - \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)R_3 \tag{1330}$$

$$= \left(R_1 - \frac{R_2}{2} - \frac{R_3}{2}\right) - j\left(\frac{\sqrt{3}}{2}R_2 - \frac{\sqrt{3}}{2}R_3\right). \tag{1331}$$

Putting this expression in the form $r\epsilon^{j\theta}$,

$$r^2 = R_1^2 + R_2^2 + R_3^2 - R_1R_2 - R_1R_3 + \frac{R_2R_3}{2} - \frac{3R_2R_3}{2} \quad (1332)$$

$$= (R_1 + R_2 + R_3)^2 - 3R_c. \quad (1333)$$

Therefore the amplitude of the double-frequency term is

$$\sqrt{3}\sqrt{\frac{(R_1 + R_2 + R_3)^2}{R_c^2} - \frac{3}{R_c}}\ E_p^2. \quad (1334)$$

2. Unbalanced Four-line Star-connected Non-reactive Load. In this case

$$a_1 = y_1 \quad (1335)$$

$$a_2 = y_2 \quad (1336)$$

$$a_3 = y_3 \quad (1337)$$

so that $\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = (y_1 + \epsilon^{j240}y_2 + \epsilon^{j120}y_3)\mathbf{W}_0.$ (1338)

Putting $y_1 = \mathrm{y}_1 = \frac{1}{R_1}$ etc., etc., (1339)

then $\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = \left(\frac{1}{R_1} + \frac{\epsilon^{j240}}{R_2} + \frac{\epsilon^{j120}}{R_3}\right)\mathbf{W}_0.$ (1340)

Expressing the operator in the form $r\epsilon^{j\theta}$, then, by analogy with the case considered above,

$$r^2 = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^2 - 3\left(\frac{1}{R_1R_2} + \frac{1}{R_2R_3} + \frac{1}{R_3R_1}\right) \quad (1341)$$

$$= \left(\frac{R_c}{R_1R_2R_3}\right)^2 - 3\left\{\frac{(R_1 + R_2 + R_3)}{R_1R_2R_3}\right\}. \quad (1342)$$

3. The General Case of the Balanced Load.—Putting

$$a_1 = a_2 = a_3 = a, \quad (1343)$$

we have from Eq. (1318)

$$\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3 = (1 + \epsilon^{j240} + \epsilon^{j120})a\mathbf{W}_0 \quad (1344)$$

$$= 0, \quad (1345)$$

as shown in Par. 82.

Thus, there is the important result that, if the load is a balanced one, the periodic component of the power is zero. In other words, the *rate at which energy is supplied to a balanced load by a symmetrical three-phase system of sine-wave e.m.fs. is constant. Such a system is, for this reason, described as a "balanced system."*

**93. Mean Power in Three-phase Systems.**—As shown in Par. 91, the equation, in general, for the mean power is

$$P = \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3). \qquad (1346)$$

If it is required to express this in terms of the loads in any given case, then

$$\mathbf{I}_1 = a_1\mathbf{E}_1 = (g_1 + js_1)\mathbf{E}_1, \qquad (1347)$$

so that

$$\mathbf{I}_1 . \mathbf{E}_1 = g_1\hat{E}^2 = 2g_1E_p^2 \qquad (1348)$$

and, similarly,

$$\mathbf{I}_2 \cdot \mathbf{E}_2 = 2g_2E_p^2 \qquad (1349)$$

$$\mathbf{I}_3 \cdot \mathbf{E}_3 = 2g_3E_p^2. \qquad (1350)$$

Therefore,

$$P = (g_1 + g_2 + g_3)E_p^2. \qquad (1351)$$

Alternatively, in terms of the line e.m.fs.

$$\mathbf{I}_1 \cdot \mathbf{E}_1 = \frac{\mathbf{I}_1}{\sqrt{3}} \cdot \epsilon^{j30}\mathbf{E}_{12} \qquad (1352)$$

$$= \frac{1}{\sqrt{3}}\,\epsilon^{-j30}\mathbf{I}_1 \cdot \mathbf{E}_{12} \qquad (1353)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - j\,\frac{1}{2}\right)(g_{12} + js_{12})\mathbf{E}_{12} \cdot \mathbf{E}_{12} \quad (1354)$$

$$= \frac{1}{\sqrt{3}}\,(\sqrt{3}g_{12} + s_{12})\frac{\hat{E}_{12}^2}{2} \qquad (1355)$$

$$= \frac{1}{\sqrt{3}}\,(\sqrt{3}g_{12} + s_{12})E_L^2 \qquad (1356)$$

and similarly for $\mathbf{I}_1 \cdot \mathbf{E}_2$ and $\mathbf{I}_3 \cdot \mathbf{E}_3$, so that

$$P = \tfrac{1}{2}\sqrt{3}\{(\sqrt{3}g_{12} + s_{12}) + (\sqrt{3}g_{23} + s_{23}) + (\sqrt{3}g_{31} + s_{31})\}E_L^2. \quad (1357)$$

To express P in terms of the constants of the loads, therefore, it is only necessary to substitute for the g and s terms the values of these quantities in terms of the constants of the loads by means of the equations already derived for the various special cases.

*Special Cases.*—1. Three-line, Star-connected Unbalanced Non-reactive Load.—Putting

$$z_1 = R_1 \qquad (1358)$$

etc., etc.

then, from Par. 84

$$g_{12} = \frac{\left(R_3 + \dfrac{R_2}{2}\right)}{R_c} \qquad (1359)$$

$$s_{12} = \frac{\sqrt{3}R_2}{2R_c} \qquad (1360)$$

where $$R_c = R_1R_2 + R_2R_3 + R_3R_1. \qquad (1361)$$

Therefore, $$\sqrt{3}g_{12} + s_{12} = \frac{\sqrt{3}(R_2 + R_3)}{R_c} \qquad (1362)$$

and similarly for $\sqrt{3}g_{23} + s_{23}$ and $\sqrt{3}g_{31} + s_{31}$, so that

$$P = \frac{(R_1 + R_2 + R_3)}{R_c} E_L^2. \qquad (1363)$$

2. Three-line Star-connected Unbalanced Reactive Load.—It is obvious from first principles that the mean power consumed in this case is zero. It will, however, confirm the validity of the general expression for the mean power to show that it leads to this result. From Par. 84, if

$$z_1 = jx_1 \qquad (1364)$$

etc., etc.

$$g_{12} = -\frac{\sqrt{3}}{2}\frac{X_2}{X_c} \qquad (1365)$$

$$s_{12} = \frac{(X_3 + \frac{1}{2}X_2)}{X_c} \qquad (1366)$$

where $$X_c = -(X_1X_2 + X_2X_3 + X_3X_1). \qquad (1367)$$

Therefore, $$\sqrt{3}g_{12} + s_{12} = \frac{(-3X_2 + 2X_3 - X_2)}{2X_c} \qquad (1368)$$

$$= \frac{(X_3 - X_2)}{X_c}, \qquad (1369)$$

so that $$P = \frac{(X_3 + X_1 + X_2) - (X_2 + X_3 + X_1)}{\sqrt{3}X_c}E_L^2 \qquad (1370)$$

$$= 0. \qquad (1371)$$

3. Three-line Star-connected Balanced Non-reactive Load.—Putting

$$R_1 = R_2 = R_3 = R_s \qquad (1372)$$

in Eq. (1363), the result for a balanced non-reactive load is

$$P = \frac{E_L^2}{R_s}. \qquad (1373)$$

4. Four-line Star-connected Unbalanced Non-reactive Load In this case

$$a_1 = \frac{1}{R_1} = g_1, \text{ etc., etc.} \qquad (1374)$$

Therefore, $$P = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) E_p^2 \qquad (1375)$$

$$= \frac{\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)}{3} E_L^2. \qquad (1376)$$

If the load is balanced this becomes

$$P = \frac{E_L^2}{R_s}, \tag{1377}$$

which is the same as Eq. (1373).

5. Delta-connected Unbalanced Non-reactive Load.—The loads being given by the admittances

$$y_{12} = G_{12} + jS_{12} \tag{1378}$$

etc., etc.,

from Par. 87

$$g_{12} = G_{12} + \tfrac{1}{2}G_{31} \tag{1379}$$

$$s_{12} = \frac{\sqrt{3}}{2} G_{31} \tag{1380}$$

therefore,

$$\sqrt{3}g_{12} + s_{12} = \sqrt{3}(G_{12} + G_{31}) \tag{1381}$$

and similarly for $\sqrt{3}g_{23} + s_{23}$ and $\sqrt{3}g_{31} + s_{31}$, so that

$$P = (G_{12} + G_{23} + G_{31})E_L^2 \tag{1382}$$

$$= \left(\frac{1}{R_{12}} + \frac{1}{R_{23}} + \frac{1}{R_{31}}\right)E_L^2, \tag{1383}$$

which can be compared with the result for a four-line star-connected load.

6. Delta-connected Balanced Non-reactive Load.—Putting

$$R_{12} = R_{23} = R_{31} = R_m \tag{1384}$$

in Eq. (1383) the result for a balanced load is

$$P = 3\,\frac{E_L^2}{R_m}.$$

**94. Mean Power in Terms of the Load Potential Differences.—** In a three-line star-connected load consisting of the impedances $z_1$, $z_2$, and $z_3$, there is in the load $z_1$ a current represented by the vector $\mathbf{I}_1$ falling through a potential represented by the vector $\mathbf{V}_1$. The mean power being consumed in this load is, therefore, $\frac{1}{2}\mathbf{I}_1 \cdot \mathbf{V}_1$. Thus, for a star-connected unbalanced load the following is an alternative expression for the mean power:

$$P = \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{V}_1 + \mathbf{I}_2 \cdot \mathbf{V}_2 + \mathbf{I}_3 \cdot \mathbf{V}_3). \tag{1385}$$

The connection between this expression and the form

$$P = \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3) \tag{1386}$$

is seen in Eqs. (1161), (1162) and (1163), by means of which Eq. (1385) can be written

$$P = -\tfrac{1}{2}\{\mathbf{I}_1 \cdot (\mathbf{E}_1 + \mathbf{V}_0) + \mathbf{I}_2 \cdot (\mathbf{E}_2 + \mathbf{V}_0) + \mathbf{I}_3 \cdot (\mathbf{E}_3 + \mathbf{V}_0)\}, \tag{1387}$$

*i.e.*,

$$P = -\tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3) - \tfrac{1}{2}(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3) \cdot \mathbf{V}_0 \tag{1388}$$

$$= -\tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3), \tag{1389}$$

since

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 0. \tag{1390}$$

The sign difference is, of course, due to the fact that Eq. (1386) refers to power given out by the supply e.m.fs., while Eq. (1385) refers to power being consumed by the loads.

**95 The Measurement of Three-phase Power.** 1. *The Three-wattmeter Method.*—Three wattmeters connected as shown in Fig.

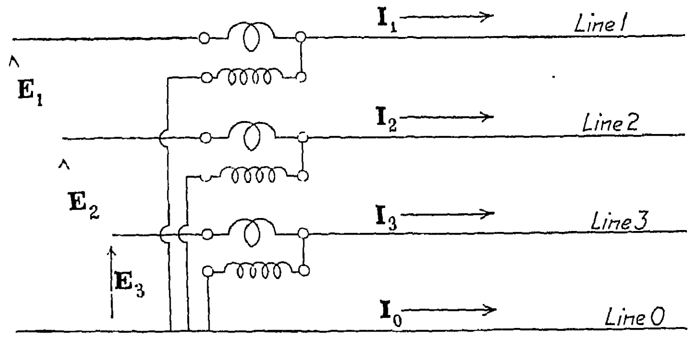


Fig. 91.

91 to the four lines of a three-phase four-line supply system will give, respectively, the values of $\frac{1}{2}\mathbf{I}_1 \cdot \mathbf{E}_1$, $\frac{1}{2}\mathbf{I}_2 \cdot \mathbf{E}_2$, and $\frac{1}{2}\mathbf{I}_3 \cdot \mathbf{E}_3$, so that the total power output of the system can be obtained by the addition of the readings of the three wattmeters (see Par. 94).

Similarly, in a three-line star-connected load with an accessible neutral point the sum of the readings of three wattmeters con-

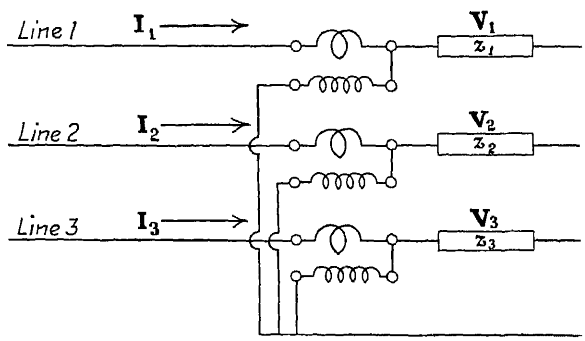


Fig. 92.

nected as shown in Fig. 92 will be the value of $\frac{1}{2}(\mathbf{I}_1 \cdot \mathbf{V}_1 + \mathbf{I}_2 \cdot \mathbf{V}$ $+ \mathbf{I}_3 \cdot \mathbf{V}_3)$, *i.e.*, of the power being consumed in the load.

*Again, if only three lines are available, or if there is no accessib neutral point in the load, an artificial neutral point can be arrang by the connection in star of three equal impedances (e.g., equ non-inductive resistances)*, as shown in Fig. 93.

In practice, however, the use of three wattmeters or the construction of an artificial neutral may be inconvenient or impossible. It is in any case unnecessary, as the following considerations will show.

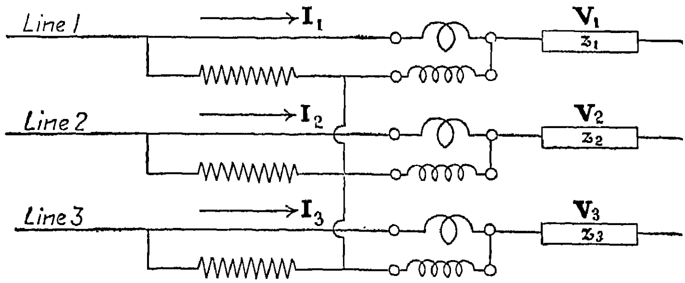


Fig. 93.

2. *Blondel's Theorem.*—Consider the system illustrated in Fig. 94, in which three wattmeters are connected with their current coils in series with the three lines, one terminal of each of the three pressure or potential coils being connected to the corresponding line, and the other three terminals of the pressure coils being connected together and either earthed, left insulated, or connected to some other point of the system which need not as yet be specified.

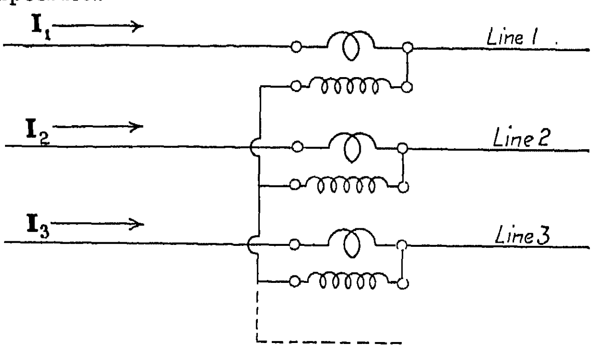


Fig. 94.

Suppose the instantaneous value of the potential of this common point with respect to the neutral point of the supply to be represented by the vector $\mathbf{E}_0$. (This vector need not be constant in magnitude or angular velocity. It can, in fact, have any value at all as long as it remains finite or zero.) Then the sum of the readings of the three wattmeters will be the mean value of the expression

$$\tfrac{1}{2}\{\mathbf{I}_1\cdot(\mathbf{E}_1-\mathbf{E}_0)+\mathbf{I}_2\cdot(\mathbf{E}_2-\mathbf{E}_0)+\mathbf{I}_3\cdot(\mathbf{E}_3-\mathbf{E}_0)\},\quad(1391)$$

*i.e.*, of

$$\tfrac{1}{2}(\mathbf{I}_1\cdot\mathbf{E}_1+\mathbf{I}_2\cdot\mathbf{E}_2+\mathbf{I}_3\cdot\mathbf{E}_3)-\tfrac{1}{2}(\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3)\cdot\mathbf{E}_0.\quad(1392)$$

Now, assuming that there is no neutral line or its equivalent in the form of earthed points on the system,

$$\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3=0\quad(1393)$$

at every instant, so that the mean value of the second part of the expression (Eq. (1392)) will, therefore, be zero, and the sum of the readings of the three wattmeters will be the mean value of

$$\tfrac{1}{2}(\mathbf{I}_1\cdot\mathbf{E}_1+\mathbf{I}_2\cdot\mathbf{E}_2+\mathbf{I}_3\cdot\mathbf{E}_3),\quad(1394)$$

which is the power output of the system.

This result is known as "Blondel's theorem." It should be noted that the proof given requires for its validity that the mean values of the scalar products concerned correctly represent the value of the mean power, and that the sum of the three-line currents is zero. It has already been shown (see Pars. 70 and 72) that the first condition is satisfied independently of the wave form of the currents and the potential differences. It is easily seen that the second condition is also satisfied independently of the wave form. Blondel's theorem, therefore, and the methods of power measurement which depend on it are independent of any assumptions with regard to the wave form. This fact is, of course, of great practical value.

3. *The Two-wattmeter Method, as a Special Case of Blondel's Theorem.*—Let the common terminal of the pressure coils be

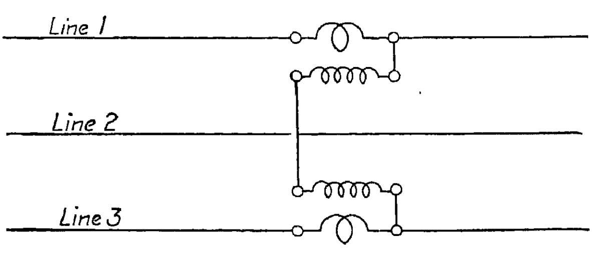


FIG. 95.

connected to line 3, as shown in Fig. 95. Then the reading of the wattmeter connected in line 3 will be zero, and the instrument can be removed without affecting the two remaining instruments, the sum of the readings of which will give the total mean power,

*i.e.,* $\frac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_{13} + \mathbf{I}_2 \cdot \mathbf{E}_{23}) = \frac{1}{2}(\mathbf{I}_1 \cdot \mathbf{E}_1 + \mathbf{I}_2 \cdot \mathbf{E}_2 + \mathbf{I}_3 \cdot \mathbf{E}_3)$ (1395)

$= \mathrm{P}.$ (1396)

This can be confirmed by putting $\mathbf{E}_0 = \mathbf{E}_3$ in Eq. (1391).

It should be noted that the sum referred to throughout in this paragraph is the *algebraic* sum of the readings obtained with the connections illustrated in the figures. Should one or more of the readings be negative (which will often be the case), then such readings must be subtracted and not added. With instruments whose scale is only on one side of the zero, a negative reading will be a deflection in the wrong direction, and, for a reading in the right direction, the connections of the current or of the pressure coils will have to be reversed. After any such reversal, therefore, the reading of the instrument must be subtracted and not added.

4. *The Power Factor of a Balanced Three-line Star-connected Load.*—For the load

$$z_1 = z_2 = z_3 = z_s \qquad (1397)$$

where $$z_s = z_s\epsilon^{j\theta_s} \qquad (1398)$$

then $$\mathbf{I}_1 = \frac{\mathbf{E}_1}{z_s}, \text{ etc., etc.,} \qquad (1399)$$

so that $\mathbf{I}_1 \cdot \mathbf{E}_1 = \mathbf{I}_2 \cdot \mathbf{E}_2 = \mathbf{I}_3 \cdot \mathbf{E}_3 = 2\mathrm{I}_L\mathrm{E}_p \cos \theta_s.$

If the power consumption of such a system is measured by the two-wattmeter method, let $\mathrm{W}_1$ and $\mathrm{W}_2$ be the readings of the instruments connected in lines 1 and 2 respectively, *i.e.,*

$$\mathrm{W}_1 = \tfrac{1}{2}\mathbf{I}_1 \cdot \mathbf{E}_{13} \qquad (1400)$$

$$\mathrm{W}_2 = \tfrac{1}{2}\mathbf{I}_2 \cdot \mathbf{E}_{23}. \qquad (1401)$$

By reference to Fig. 76 it will be seen that

$$\mathbf{E}_{13} = -\mathbf{E}_{31} = -\sqrt{3}\epsilon^{-j30}\mathbf{E}_3 = \sqrt{3}\epsilon^{j30}\mathbf{E}_1 \qquad (1402)$$

$$\mathbf{E}_{23} = +\sqrt{3}\epsilon^{-j30}\mathbf{E}_2. \qquad (1403)$$

Therefore, $$\mathrm{W}_1 = \mathbf{I}_1 \cdot \sqrt{3}\epsilon^{j30}\mathbf{E}_1 \qquad (1404)$$

$$= \sqrt{3}\hat{\mathrm{I}}_1\hat{\mathrm{E}}_1 \cos (\theta_s - 30) \qquad (1405)$$

and $$\mathrm{W}_2 = \mathbf{I}_2 \cdot \sqrt{3}\epsilon^{-j30}\mathbf{E}_2 \qquad (1406)$$

$$= \sqrt{3}\hat{\mathrm{I}}_2\hat{\mathrm{E}}_2 \cos (\theta_s + 30). \qquad (1407)$$

Therefore, $$\mathrm{W}_1 + \mathrm{W}_2 = 2\sqrt{3}\hat{\mathrm{I}}_1\hat{\mathrm{E}}_1 \cos \theta_s \cos 30 \qquad (1408)$$

$$= 3.2 \cdot \mathrm{I}_L\mathrm{E}_p \cos \theta_s \qquad (1409)$$

and $$\mathrm{W}_1 - \mathrm{W}_2 = 2\sqrt{3}\hat{\mathrm{I}}_2\hat{\mathrm{E}}_2 \sin \theta_s \sin 30 \qquad (1410)$$

$$= 2\sqrt{3}\,\hat{\mathrm{I}}_L\hat{\mathrm{E}}_p \sin \theta_s, \qquad (1411)$$

so that the power factor of the load $z_s$ is given by

$$\cos \theta_s = \cos \left\{\tan^{-1} \sqrt{3}\frac{\mathrm{W}_1 - \mathrm{W}_2}{\mathrm{W}_1 + \mathrm{W}_2}\right\}. \qquad (1412)$$

**96. The General Theory of Symmetrical Polyphase Systems of E.M.Fs. of Any Wave Form.**—The general definition of such a system has already been given in Par. 74. Some of the more important features of the general theory of such a system will now be considered, with special reference to the three-phase system of e.m.fs. of irregular wave form.

*Note.*—In the following paragraphs, subscripts denoting phase number or any related specification will be enclosed in brackets in order to distinguish them from subscripts denoting harmonic order.

1. *Graphical Representation.*—In general, the angle between the vectors representing successive members of an m-phase system of e.m.fs. of irregular wave form is not constant but is a periodic function of time. It is impossible, therefore, to represent such a system by a figure of constant shape. If, however, the e.m.fs. concerned are of approximately sine-wave form, then

$$\mathbf{E}_{(1)} = \mathbf{E} \tag{1413}$$

$$\mathbf{E}_{(2)} = \epsilon^{j\phi}\mathbf{E} \tag{1414}$$

$$\mathbf{E}_{(3)} = \epsilon^{j2\phi}\mathbf{E} \tag{1415}$$

and for the $r^{th}$ phase

$$E_{(r)} = \epsilon^{j(r-1)\phi}E$$

where

$$\phi = \frac{2\pi}{m}, \tag{1416}$$

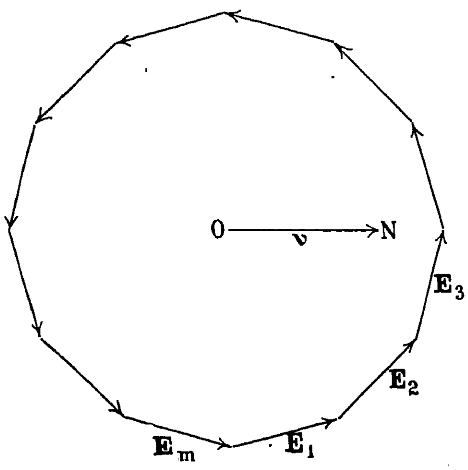


Fig. 96

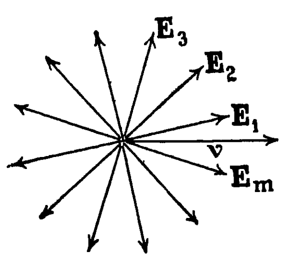


Fig. 97.

so that the members of a sine wave system can be represented by the sides of a regular polygon of m sides, as shown in Fig. 96, or by the star-shaped Fig. 97.

2 *The Sum of the E.M.Fs. of a Symmetrical m-phase System.*—Let $\mathbf{E}_{(s)}$ be the vector which represents this sum, *i.e.*,

$$\mathbf{E}_{(s)} = \sum_{r=1}^{r=m} \mathbf{E}_{(1)}. \qquad (1417)$$

If the $n^{th}$ harmonic of $\mathbf{E}_{(s)}$ be represented by $\mathbf{E}_{(s)n}$, then (see Par. 74)

$$\mathbf{E}_{(s)n} = (1 + \epsilon^{jn\phi} + \epsilon^{j2n\phi} + \ . \ . \ . + \epsilon^{j(m-1)n\phi})\mathbf{E}_n. \qquad (1418)$$

Operating on each side of this equation with $\epsilon^{jn\phi}$,

$$\epsilon^{jn\phi}\mathbf{E}_{(s)n} = (\epsilon^{jn\phi} + \epsilon^{j2n\phi} + \ . \ . \ . \ \epsilon^{jmn\phi})\mathbf{E}_n \qquad (1419)$$

and, by subtraction,

$$(1 - \epsilon^{jn\phi})\mathbf{E}_{(s)n} = (1 - \epsilon^{jmn\phi})\mathbf{E}_n. \qquad (1420)$$

Therefore,

$$\mathbf{E}_{(s)n} = \left(\frac{1 - \epsilon^{jmn\phi}}{1 - \epsilon^{jn\phi}}\right)\mathbf{E}_n \qquad (1421)$$

and, since

$$\phi = \frac{2\pi}{m}, \qquad (1422)$$

this can be put in the form

$$\mathbf{E}_{(s)m} = \left(\frac{1 - \epsilon^{j2n\pi}}{1 - \epsilon^{j\frac{2n\pi}{m}}}\right)\mathbf{E}_n \qquad (1423)$$

Now

$$(1 - \epsilon^{j2n\pi}) = 0, \qquad (1424)$$

so that, in general,

$$\mathbf{E}_{(s)n} = 0. \qquad (1425)$$

If, however, n = km, where k is any positive integer, $(1 - \epsilon^{j\frac{2n\pi}{m}})$ also equals zero, so that in all such cases, *i.e.*, for all the harmonics whose order is a multiple of m,

$$\mathbf{E}_{(s)n} = \frac{0}{0} \qquad (1426)$$

and appears to be indeterminate. Its value can, however, be found as usual by differentiating the numerator and the denominator of the operational expression in Eq. (1423) with respect to n and then putting n = km in the result.

Thus

$$\mathrm{Lt}\left(\frac{1 - \epsilon^{j2n\pi}}{1 - \epsilon^{j\frac{2n\pi}{m}}}\right) = \mathrm{Lt}\,\frac{m\epsilon^{j2n\pi}}{\epsilon^{j\frac{2n\pi}{m}}} \qquad (1427)$$

$$= m \text{ when } n = km. \qquad (1428)$$

Therefore, $$\mathbf{E}_{(s)n} = 0 \text{ in general} \qquad (1429)$$

$$= m\mathbf{E}_n \text{ when } n = km \qquad (1430)$$

and, since $$\mathbf{E}_{(s)} = \sum_{n=1}^{n=\infty} \mathbf{E}_{(s)n}, \qquad (1431)$$

the result is $$\mathbf{E}_{(s)} = m \sum \mathbf{E}_n \text{ where } n = m, 2m, 3m, \text{ etc.} \qquad (1432)$$

*Thus, the sum of the e.m.fs. of a symmetrical* m*-phase system is an e.m.f. consisting of* m *times those harmonics of the first phase whose order is a multiple of* m.

*In particular, the sum of the e.m.fs. of a three-phase system consists of three times those harmonics of the first phase whose order is a multiple of* 3.

3. *The Interconnection of the* m*-phase System.*—An m-phase system can be interconnected in either of the ways already described in relation to the three-phase system. The result of the interconnection is, however, different from the corresponding results in the simpler case.

(*a*) Star Interconnection.—For the e.m.f. between lines 1 and 2

$$\mathbf{E}_{(12)} = \mathbf{E}_{(1)} - \mathbf{E}_{(2)} \qquad (1433)$$

and for the $n^{th}$ harmonics of these e.m.fs.

$$\mathbf{E}_{(12)n} = \mathbf{E}_{(1)n} - \mathbf{E}_{(2)n} \qquad (1434)$$

$$= (1 - \epsilon^{jn\phi})\mathbf{E}_{(1)n}. \qquad (1435)$$

Putting this operator in the form $r\epsilon^{j\theta}$,

$$r^2 = (1 - \cos n\phi)^2 + \sin^2 n\phi \qquad (1436)$$

$$= 2(1 - \cos n\phi) \qquad (1437)$$

$$= 2.2 \sin^2 \frac{n\phi}{2}. \qquad (1438)$$

Therefore, $$r = 2 \sin \frac{n\phi}{2} \qquad (1439)$$

$$= 2 \sin \frac{n\pi}{m}. \qquad (1440)$$

Also, $$\sin\theta = \frac{-\sin n\phi}{\left(2 \sin \frac{n\phi}{2}\right)} \qquad (1441)$$

$$= -\cos \frac{n\phi}{2} \qquad (1442)$$

and
$$\cos\theta = \frac{(1 - \cos n\phi)}{2\sin\frac{n\phi}{2}} \tag{1443}$$

$$= \sin\frac{n\phi}{2} \tag{1444}$$

so that
$$\theta = -\frac{\pi}{2} + \frac{n\phi}{2}. \tag{1445}$$

Equation (1435), therefore, becomes

$$\mathbf{E}_{(12)n} = 2\sin\frac{n\pi}{m}\epsilon^{-j\left(\frac{\pi}{2} - \frac{n\phi}{2}\right)}\mathbf{E}_{(1)n} \tag{1446}$$

and the line e.m.f. for lines 1 and 2 is

$$\mathbf{E}_{(12)} = 2\sum_{n=1}^{n=\infty}\sin\frac{n\pi}{m}\epsilon^{-j\left(\frac{\pi}{2} - \frac{n\phi}{2}\right)}\mathbf{E}_{(1)n}. \tag{1447}$$

The remaining line e.m.fs. of the system can be derived from

$$\mathbf{E}_{(23)} = \sum_{n=1}^{n=\infty}\epsilon^{jn\phi}\mathrm{E}_{(12)n} \tag{1448}$$

$$\mathbf{E}_{(34)} = \sum_{n=1}^{n=\infty}\epsilon^{j2n\phi}\mathbf{E}_{(12)n} \tag{1449}$$

etc., etc.

It should be noted that the wave form of the line e.m.fs. is different from that of the phase e.m.fs., owing to the absence of certain harmonics from the former. For all values of n such that $n = km$ where k is any positive integer, Eq. (1440) gives

$$r = 0 \tag{1450}$$

*Thus, all the harmonics the orders of which are multiples of the numbers of phases are absent from the line e.m.fs. of a star-connected system.* In particular, the third, sixth, ninth, etc. harmonics will be absent from the line e.m.fs. of a star-connected three-phase system. Since in nearly all practical cases only the odd harmonics are present in the wave form, this means that the first harmonic present in the wave form of the line e.m.f. of such a system will be the fifth and, since, in general, the amplitude of the harmonics decreases rapidly with their order, this implies that the assumption of sine-wave form for the line e.m.fs. will usually result in a fairly high degree of accuracy.

It should further be noted that since

$$\mathbf{E}_{(12)} = \mathbf{E}_{(1)} - \mathbf{E}_{(2)} \qquad (1451)$$
$$\mathbf{E}_{(23)} = \mathbf{E}_{(2)} - \mathbf{E}_{(3)} \qquad (1452)$$
$$\mathbf{E}_{(34)} = \mathbf{E}_{(3)} - \mathbf{E}_{(4)} \qquad (1453)$$

etc., etc.

$$\mathbf{E}_{(12)} + \mathbf{E}_{(23)} + \mathbf{E}_{(34)} + \mathbf{E}_{(45)} + \text{etc., etc.} = 0$$

quite independently of any assumptions with regard to wave form.

(*b*) Ring Interconnection.—In ring interconnection the phase windings form a closed path for the e.m.f. produced by the addition of all the phase e m.fs. It has been shown that this resultant e.m.f. is equal to m times the sum of the harmonics of the first phase whose order is a multiple of m. In general, the closed circuit in which this e.m.f. operates will have a low impedance, so that the amplitude of the circulating currents may be considerable. This fact has to be taken into account in the design of ring-connected windings, and constitutes an inherent disadvantage of this type of connection as compared with star interconnection.

The short-circuit path provided by the phase windings removes from the line pressures just those harmonics which would otherwise appear in the sum of the line pressures. *Therefore, both with star and with ring interconnection, harmonics whose order is a multiple of the number of phases will be absent from the line e.m.fs., and the sum of the line e.m.fs. will be zero.*

4. *Polyphase Loads.*—The loads to which polyphase systems of e.m.fs. are applied can be similarly interconnected in star or in ring. The current and potential relationships for any given case can be obtained in a precisely similar manner to that employed in the three-phase system already considered, each harmonic being taken separately and the total result obtained by the summation of the individual results for the separate harmonics.

As an example, take a three-line star-connected unbalanced three-phase load consisting of impedances whose values corresponding to the frequency of the $n^{th}$ harmonic will be written $z_{(1)n}$, $z_{(2)n}$, and $z_{(3)n}$.

By analogy with Eqs. (1169), (1170) and (1171) of this chapter, the expressions for the $n^{th}$ harmonic are:

$$\mathbf{I}_{(1)n}z_{(1)n} - \mathbf{I}_{(2)n}z_{(2)n} = \mathbf{E}_{(12)n} \qquad (1454)$$
$$\mathbf{I}_{(2)n}z_{(2)n} - \mathbf{I}_{(3)n}z_{(3)n} = \mathbf{E}_{(23)n} \qquad (1455)$$
$$\mathbf{I}_{(3)n}z_{(3)n} - \mathbf{I}_{(1)n}z_{(1)n} = \mathbf{E}_{(31)n} \qquad (1456)$$

and for the neutral point of the load

$$\mathbf{I}_{(1)n} + \mathbf{I}_{(2)n} + \mathbf{I}_{(3)n} = 0. \tag{1457}$$

As in Eq. (1180), the solution for $\mathbf{I}_{(1)n}$ will be

$$\mathbf{I}_{(1)n} = \frac{z_{(3)n}\mathbf{E}_{(12)n} - z_{(2)n}\mathbf{E}_{(31)n}}{z_{(1)n}z_{(2)n} + z_{(2)n}z_{(3)n} + z_{(3)n}z_{(1)n}} \tag{1458}$$

and similarly for $\mathbf{I}_{(2)n}$ and $\mathbf{I}_{(3)n}$.

The total currents $\mathbf{I}_{(1)}$, $\mathbf{I}_{(2)}$, $\mathbf{I}_{(3)}$, will, therefore, be

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \mathbf{I}_{(1)n} = \sum_{n=1}^{n=\infty} \frac{(z_{(3)n}\mathbf{E}_{(12)n} - z_{(2)n}\mathbf{E}_{(31)n})}{z_{(c)n}} \tag{1459}$$

$$\mathbf{I}_{(2)} = \sum_{n=1}^{n=\infty} \mathbf{I}_{(2)n} = \sum_{n=1}^{n=\infty} \frac{(z_{(1)n}\mathbf{E}_{(23)n} - z_{(3)n}\mathbf{E}_{(12)n})}{z_{(c)n}} \tag{1460}$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \mathbf{I}_{(3)n} = \sum_{n=1}^{n=\infty} \frac{(z_{(2)n}\mathbf{E}_{(31)n} - z_{(1)n}\mathbf{E}_{(23)n})}{z_{(c)n}} \tag{1461}$$

where

$$z_{(c)n} = z_{(1)n}z_{(2)n} + z_{(2)n}z_{(3)n} + z_{(3)n}z_{(1)n}.$$

For reasons already given, the above currents will not contain harmonics whose order is a multiple of 3.

If it is desired to state $\mathbf{I}_{(1)}$ in terms of $\mathbf{E}_{(12)}$ only, since

$$\mathbf{E}_{(31)n} = \epsilon^{j2n\phi}\mathbf{E}_{(12)n} \tag{1462}$$

then

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \frac{z_{(3)n} - \epsilon^{j2n\phi}z_{(2)n}}{z_{(c)n}}\mathbf{E}_{(12)n}. \tag{1463}$$

Similarly,

$$\mathbf{I}_{(2)} = \sum_{n=1}^{n=\infty} \frac{z_{(1)n} - \epsilon^{j2n\phi}z_{(3)n}}{z_{(c)n}}\mathbf{E}_{(23)n} \tag{1464}$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \frac{z_{(2)n} - \epsilon^{j2n\phi}z_{(1)n}}{z_{(c)n}}\mathbf{E}_{(31)n}. \tag{1465}$$

If the loads are balanced, *i.e.*, if

$$z_{(1)n} = z_{2(n)} = z_{(3)n} = z_{(s)n}, \tag{1466}$$

then

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} (1 - \epsilon^{j2n\phi})\frac{\mathbf{E}_{(12)n}}{3z_{(s)n}} \tag{1467}$$

$$= \sum_{n=1}^{n=\infty} (1 - \epsilon^{j2n\phi})(1 - \epsilon^{jn\phi})\frac{\mathbf{E}_{(1)n}}{3z_{(s)n}} \tag{1468}$$

$$= \sum_{n=1}^{n=\infty} (1 - \epsilon^{jn\phi} - \epsilon^{j2n\phi} + \epsilon^{j3n\phi})\frac{\mathbf{E}_{(1)n}}{3z_{(s)n}} \tag{1469}$$

$$= \sum_{n=1}^{n=\infty} \left\{3 - (\epsilon^{jn\phi} + \epsilon^{j2n\phi} + \epsilon^{j3n\phi})\right\}\frac{\mathbf{E}_{(1)n}}{3z_{(s)n}}. \tag{1470}$$

But, as shown in Par. 96,

$$\epsilon^{jn\phi} + \epsilon^{j2n\phi} + \epsilon^{j3n\phi} = 0 \qquad (1471)$$

unless n = 3k, all of which cases are already excluded.

Therefore,

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(1)n}}{z_{(s)n}} \qquad (1472)$$

$$\mathbf{I}_{(2)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(2)n}}{z_{(s)n}} \qquad (1473)$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(3)n}}{z_{(s)n}}, \qquad (1474)$$

there being in none of these currents any harmonics of orders which are a multiple of 3.

(*a*) Four-line Star-connected Load.—The analysis of this case is comparatively simple, since

$$\mathbf{V}_{(1)} = \mathbf{E}_{(1)} \qquad (1475)$$

etc., etc.,

so that

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(1)n}}{z_{(1)n}} \qquad (1476)$$

$$\mathbf{I}_{(2)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(2)n}}{z_{(2)n}} \qquad (1477)$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{3(n)}}{z_{(3)n}}. \qquad (1478)$$

For the current in the fourth line,

$$\mathbf{I}_{(0)} = -(\mathbf{I}_{(1)} + \mathbf{I}_{(2)} + \mathbf{I}_{(3)}) \qquad (1479)$$

$$= \sum_{n=1}^{n=\infty} -\left(\frac{\mathbf{E}_{(1)n}}{z_{(1)n}} + \frac{\mathbf{E}_{(2)n}}{z_{(2)n}} + \frac{\mathbf{E}_{(3)n}}{z_{(3)n}}\right) \qquad (1480)$$

$$= -\sum_{n=1}^{n=\infty} \left(\frac{1}{z_{(1)n}} + \frac{\epsilon^{jn\phi}}{z_{(2)n}} + \frac{\epsilon^{j2n\phi}}{z_{(3)n}}\right)\mathbf{E}_{(1)n}. \qquad (1481)$$

If the loads are balanced, then for the line currents

$$\mathbf{I}_{(1)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(1)n}}{z_{(s)n}} \qquad (1482)$$

$$\mathbf{I}_{(2)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(2)n}}{z_{(s)n}} \qquad (1483)$$

$$\mathbf{I}_{(3)} = \sum_{n=1}^{n=\infty} \frac{\mathbf{E}_{(3)n}}{z_{(s)n}}. \qquad (1484)$$

This appears to be the same as the result for the corresponding three-line case, but it must be remembered that in the present case there is nothing to prevent the existence of the harmonics whose order is a multiple of 3.

For the current in the fourth line,

$$I_{(0)} = \sum_{n=1}^{n=\infty} (1 + \epsilon^{jn\phi} + \epsilon^{j2n\phi}) \frac{E_{(1)n}}{z_{(s)n}} \tag{1485}$$

and, as shown in Par. 96, (2),

$$1 + \epsilon^{jn\phi} + \epsilon^{j2n\phi} = 0 \text{ in general} \tag{1486}$$

$$= 3 \text{ when } n = 3, 6, 9, \text{ etc.,} \tag{1487}$$

so that

$$I_{(0)} = 3 \sum \frac{E_{(1)n}}{z_{(s)n}} \tag{1488}$$

where $n = 3, 6, 9$, etc., etc., so that, even with a balanced load, there is still a current flowing in the neutral line consisting of harmonics whose order is a multiple of 3.

(*b*) Delta-connected Loads.—As in the sine-wave case already considered,

$$I_{(1)} = I_{(12)} - I_{(31)} \tag{1489}$$

etc., etc.

Also

$$I_{(12)} = \sum_{n=1}^{n=\infty} \frac{E_{(12)n}}{z_{(12)n}} \tag{1490}$$

$$I_{(23)} = \sum_{n=1}^{n=\infty} \frac{E_{(23)n}}{z_{(23)n}} \tag{1491}$$

$$I_{(31)} = \sum_{n=1}^{n=\infty} \frac{E_{(31)n}}{z_{(31)n}}, \tag{1492}$$

so that

$$I_{(1)} = \sum_{n=1}^{n=\infty} \left( \frac{E_{(12)n}}{z_{(12)n}} - \frac{E_{(31)n}}{z_{(31)n}} \right) \tag{1493}$$

and similarly for $I_{(2)}$ and $I_{(3)}$.

In terms of $E_{(12)}$ only Eq. (1493) can be written

$$I_{(1)} = \sum_{n=1}^{n=\infty} \left( \frac{1}{z_{(12)n}} - \frac{\epsilon^{j2n\phi}}{z_{(31)n}} \right) E_{(12)n} \tag{1494}$$

and the corresponding expressions for $I_{(2)}$ and $I_{(3)}$ can be written by symmetry.

If the load is balanced, then putting

$$z_{(12)n} = z_{(23)n} = z_{(31)n} = z_{(m)n} \tag{1495}$$

the result is

$$I_{(1)} = \sum_{n=1}^{n=\infty} (1 - \epsilon^{j2n\phi}) \frac{E_{(12)n}}{z_{(m)n}} \tag{1496}$$

or, in terms of $\mathbf{E}_{(1)}$,

$$\mathbf{I}_{(1)} = (1 - \epsilon^{j2n\phi})(1 - \epsilon^{jn\phi})\frac{\mathbf{E}_{(1)n}}{z_{(m)n}} \tag{1497}$$

and, as shown in Eqs. (1468) to (1470), the operator product equals 3 in general and 0 when n = 3, 6, 9, etc., so that

$$\mathbf{I}_{(1)} = 3\sum_{n=1}^{n=\infty}\frac{E_{(1)n}}{z_{(m)n}} \tag{1498}$$

$$\mathbf{I}_{(2)} = 3\sum_{n=1}^{n=\infty}\frac{E_{(2)n}}{z_{(m)n}} \tag{1499}$$

$$\mathbf{I}_{(3)} = 3\sum_{n=1}^{n=\infty}\frac{E_{(3)n}}{z_{(m)n}}, \tag{1500}$$

all harmonics whose order is a multiple of 3 being excluded from the series.

5. *Equivalence Conditions of Star- and Ring-connected Loads.*—By a comparison of the line current equations for star- and ring-connected loads, it is seen that loads consisting of the impedances $z_{(1)}$, $z_{(2)}$, $z_{(3)}$, connected in lines 1, 2, and 3 (there being no neutral line), will be exactly equivalent to loads $z_{(12)}$, $z_{(23)}$, and $z_{(31)}$ connected in ring between lines 1 and 2, 2 and 3, 3 and 1, provided

$$\frac{1}{z_{(23)n}} = \frac{z_{(1)n}}{z_{(c)n}} \tag{1501}$$

$$\frac{1}{z_{(31)n}} = \frac{z_{(2)n}}{z_{(c)n}} \tag{1502}$$

$$\frac{1}{z_{(12)n}} = \frac{z_{(3)n}}{z_{(c)n}} \tag{1503}$$

where $$z_{(c)n} = z_{(1)n}z_{(2)n} + z_{(2)n}z_{(3)n} + z_{(3)n}z_{(1)n}. \tag{1504}$$

If the loads be expressed as admittances, the equivalence condition takes the form

$$y_{(12)n} = \frac{(y_{(1)n}y_{(2)n})}{y_{(c)n}} \tag{1505}$$

$$y_{(23)n} = \frac{(y_{(2)n}y_{(3)})}{y_{(c)n}} \tag{1506}$$

$$y_{(31)n} = \frac{(y_{(3)n}y_{(1)n})}{y_{(c)n}} \tag{1507}$$

where $$y_{(c)n} = y_{(1)n} + y_{(2)n} + y_{(3)n}. \tag{1508}$$

As in Par. 89 the equivalence condition can be put in the alternative form

$$y_{(1)n} = y_{(12)n} + y_{(31)n} + \frac{(y_{(12)n}y_{(31)n})}{y_{(23)n}} \qquad (1509)$$

$$y_{(2)n} = y_{(23)n} + y_{(12)n} + \frac{(y_{(23)n}y_{(12)n})}{y_{(31)n}} \qquad (1510)$$

$$y_{(3)n} = y_{(31)n} + y_{(23)n} + \frac{(y_{(31)n}y_{(23)n})}{y_{(12)n}}. \qquad (1511)$$

In practice it is only in certain simple cases that it will be possible to fulfil these conditions for the whole range of frequencies. If, for instance, the loads are pure inductances or pure resistances or pure capacities, the arrangement of conductors which satisfies the equivalence condition is independent of frequency and will, therefore, hold good for all the harmonics and for the fundamental.

There is no practical difficulty in balanced loads. Putting

$$y_{(1)n} = y_{(2)n} = y_{(3)n} = y_{(s)n} \qquad (1512)$$

$$y_{(12)n} = y_{(23)n} = y_{(31)n} = y_{(m)n} \qquad (1513)$$

the conditions reduce to

$$3y_{(m)n} = y_{(s)n}, \qquad (1514)$$

which would be comparatively easy to satisfy in any given case.

6. *Power in Three-phase Systems of General Wave Form.*—The extension of the analysis given in Pars. 91 and 92 to the harmonics of a three-phase system does not present any great difficulty, but the subject is of little practical interest and will not, therefore, be considered in detail.

The discussion in Par. 95 of the methods of measurement of the mean power in three-phase systems depends on assumptions which are equally valid for currents and potentials of general wave form and of sine-wave form, and will, therefore, apply equally well to the case of three-phase power of general wave form. This is due to the fact, proved in Par. 72, that, whatever may be the wave form of a current represented by the vector **I** and a potential difference represented by the vector **E**, the average value over a period of $\frac{1}{2}\mathbf{I} \cdot \mathbf{E}$ will be the average value of the power supplied by the current **I** in falling through the potential **E**, and the magnitude of this quantity can be determined in the usual way by connecting a wattmeter so that the current **I** passes through its current coils, its potential coils being connected across the potential **E**.

## EXAMPLES

*Note.*—Any symbols used in these examples will have the same significance as in the preceding text.

1. Three loads, $z_1$, $z_2$, and $z_3$, consist respectively of a pure resistance of 100 ohms, a pure resistance of 50 ohms in series with an inductance of .239 henries, and a pure resistance of 75 ohms in series with an inductance of of .159 henries. The supply is a three-phase, three or four line, the line voltage being 70.7 volts and the frequency 50 p.p.s.
   (*a*) The load being connected in star, three line, calculate:
      (1) The line currents as vectors at the instant $t = 0$, in terms of the unit vector $\mathbf{v}$, being given that $\mathbf{E}_{12} = \hat{E}_{12}\mathbf{v}$ when $t = 0$.
      (2) The corresponding scalar expressions for the instantaneous values of the line currents at any instant t.
      (3) The potential of the neutral point with respect to that of the neutral point of the supply, as a vector in terms of $\mathbf{v}$ at the instant $t = 0$.
      (4) The scalar expression for the instantaneous value of the relative potential of the neutral point at any instant t.
   (*b*) The load being connected in star, four line, calculate:
      (1) The line currents and the current in the neutral line as vectors in terms of $\mathbf{v}$ at the instant $t = 0$.
      (2) The corresponding expressions in scalar form for the instantaneous values of the line currents and the current in the neutral line at any instant t.
      (3) Verify that the current in the neutral line is the same as would be produced by an e.m.f. identical in magnitude and phase with the relative potential of the neutral point of the load in the three-line case acting in a circuit consisting of the above three loads connected in parallel.
   (*c*) The load being connected in $\Delta$, *i.e.*, $z_{12} = 100$ ohms, etc., calculate:
      (1) The line currents as vectors in terms of $\mathbf{v}$ at the instant $t = 0$.
      (2) The corresponding scalar expressions for the instantaneous values of the line currents at any instant t.
      (3) The currents through the loads, as in $(c - 1)$.
      (4) The currents through the loads, as in $(c - 2)$.
   (*d*) The above load being connected in three line, star, find the equivalent $\Delta$-connected load. Express the results as admittances in the form $\mathrm{r}\epsilon^{j\theta}$.
   (*e*) The above load being connected in $\Delta$, *i.e.*, $z_{12} = 100$ ohms, etc., find the equivalent star-connected load, the results being expressed as admittances in the form $\mathrm{r}\epsilon^{j\theta}$.
2. Find the mean power consumed in the load in each of the alternative connections specified above, *i.e.*,
   (*a*) Star, three line.
   (*b*) Star, four line.
   (*c*) $\Delta$.

   In (*a*) and (*b*) calculate the mean power by the two-wattmeter method formula (see Eq. (1396)) and also by the summation of $I^2R$, showing that the results obtained are in agreement.

**3.** Each arm of a balanced star-connected three-phase load consists of a pure resistance of 100 ohms in parallel with a pure inductance of .159 henries. The supply frequency is 50 p.p.s., and the line e.m.f. is 500 volts.

(a) Representing the line e.m.f. $\mathbf{E}_{12}$ by the vector $500\sqrt{2}\mathbf{v}$, at the instant $t = 0$, express the line currents as vectors in terms of $\mathbf{v}$ at the instant $t = 0$.

(b) Give the corresponding scalar expressions for the line currents at any instant t.

(c) The power input to the load is measured by the two-wattmeter method. Calculate $W_1$ and $W_2$, the readings of the two wattmeters, giving each its appropriate sign, and hence confirm that $\theta$, the phase angle of each of the three equal loads, is given by

$$\theta = \tan^{-1}\left\{\sqrt{3}\left(\frac{W_{(1)} - W_{(2)}}{W_{(1)} + W_{(2)}}\right)\right\}.$$

**4.** The loads specified in Example 3 are connected in star with a line connecting the neutral points of the load and the supply. Given that $\mathbf{E}_{(1)}$ is represented at the instant $t = 0$ by the vector $100\mathbf{v}$ (the supply frequency being 50 p.p.s.) and that the phase e.m.f. has a third harmonic in phase with the fundamental and 15 per cent of it in magnitude, calculate the current in the neutral line. Express the result as a vector in terms of $\mathbf{v}$ at the instant $t = 0$, and also in scalar form for any instant t.

## ANSWERS TO EXAMPLES

**1.** (a) (1) $\mathbf{I}_{(1)} = .544\epsilon^{j16.2^\circ}\mathbf{v}$
$\mathbf{I}_{(2)} = .540\epsilon^{j104.9^\circ}\mathbf{v}$
$\mathbf{I}_{(3)} = .788\epsilon^{j239.5^\circ}\mathbf{v}$.

(2) $i_{(1)} = .544\cos(100\pi t + 16.2^\circ)$ amperes
$i_{(2)} = .540\cos(100\pi t + 104.9^\circ)$ amperes
$i_{(3)} = .788\cos(100\pi t + 239.5^\circ)$ amperes.

(3) $\mathbf{V}_0 = -13.80\epsilon^{j106.18^\circ}\mathbf{v}$.

(4) $v_0 = -13.80\cos(100\pi t + 106.18^\circ)$ volts.

(b) (1) $\mathbf{I}_{(1)} = .578\epsilon^{j30^\circ}\mathbf{v}$
$\mathbf{I}_{(2)} = .638\epsilon^{j93.7^\circ}\mathbf{v}$; $\mathbf{I}_0 = -.41\epsilon^{j74.4^\circ}\mathbf{v}$
$\mathbf{I}_{(3)} = .638\epsilon^{j236.3^\circ}\mathbf{v}$.

(2) $i_{(1)} = .578\cos(100\pi t + 30^\circ)$ amperes
$i_{(2)} = .638\cos(100\pi t + 93.7^\circ)$ amperes
$i_{(3)} = .638\cos(100\pi t + 236.3^\circ)$ amperes
$i_{(0)} = -.41\cos(100\pi t + 74^\circ 4)$ amperes.

(3) Admittance of the three loads in parallel
$y_{(c)} = .0295\epsilon^{-j31^\circ}$
$\mathbf{V}_0 = -13.80\epsilon^{j106.18^\circ}\mathbf{v}$
$y_{(c)}\mathbf{V}_{(0)} = -.41\epsilon^{j75^\circ}\mathbf{v}$.

(c) (1) $\mathbf{I}_{(1)} = 2.05\epsilon^{j13.8^\circ}\mathbf{v}$
$\mathbf{I}_{(2)} = 1.11\epsilon^{j152.7^\circ}\mathbf{v}$
$\mathbf{I}_{(3)} = 2.09\epsilon^{j225^\circ}\mathbf{v}$.

(2) $i_{(1)} = 2.05\cos(100\pi t + 13.8^\circ)$ amperes
$i_{(2)} = 1.11\cos(100\pi t + 152.7^\circ)$ amperes
$i_{(3)} = 2.09\cos(100\pi t + 225^\circ)$ amperes.

(3) $\mathbf{I}_{(12)} = \mathbf{v}$
$\mathbf{I}_{(23)} = 1.104\epsilon^{j63.7°}\mathbf{v}$
$\mathbf{I}_{(31)} = 1.104\epsilon^{j206.3°}\mathbf{v}$.
(4) $i_{(12)} = \cos(100\pi t + 0°)$ amperes
$i_{(23)} = 1.104 \cos(100\pi t + 63.7°)$ amperes
$i_{(31)} = 1.104 \cos(100\pi t + 206.3°)$ amperes.
(*d*) $.00374\epsilon^{-j25.12°}$
$.00418\epsilon^{-j58.82°}$
$.00374\epsilon^{-j2.52°}$.
(*e*) $.0285\epsilon^{-j4.6°}$
$.0285\epsilon^{-j27.2°}$
$.0315\epsilon^{-j60.9°}$.

**2.** (*a*) 45.37 watts.
(*b*) 42.14 watts.
(*c*) 126.2 watts.

**3.** (*a*) $\mathbf{I}_{(1)} = 9.15\epsilon^{-j33.45°}\mathbf{v}$
$\mathbf{I}_{(2)} = 9.15\epsilon^{j86.55°}\mathbf{v}$
$\mathbf{I}_{(3)} = 9.15\epsilon^{j206.55°}\mathbf{v}$.
(*b*) $i_{(1)} = 9.15 \cos(100\pi t - 33.45°)$ amperes
$i_{(2)} = 9.15 \cos(100\pi t + 86.55°)$ amperes
$i_{(3)} = 9.15 \cos(100\pi t + 206.55°)$ amperes.
(*c*) $W_{(1)} = -195$ watts
$W_{(2)} = 2{,}700$ watts

$$\theta = \tan^{-1}\left\{-\sqrt{3}\left(\frac{2895}{2505}\right)\right\} = -\tan^{-1}2.0.$$

**4.** $\mathbf{I}_{(0)3} = .542\epsilon^{-j33.66°}\mathbf{v}$
$i_{(0)3} = .542 \cos(300\pi t - 33.66°)$ amperes.

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